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THE FINITISTIC DIMENSION CONJECTURE AND RELATIVELY PROJECTIVE MODULES

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The famous finitistic dimension conjecture says that every finite-dimensional $\mathbb{K}$-algebra over a field $\mathbb{K}$ should have finite finitistic dimension. This conjecture is equivalent to the following statement: If B is a subalgebra of a finite-dimensional $\mathbb{K}$-algebra A such that the radical of B is a left ideal in A, and if A has finite finitistic dimension, then B has finite finitistic dimension. In the paper, we shall work with a more general setting of Artin algebras. Let B be a subalgebra of an Artin algebra A such that the radical of B is a left ideal in A. (1) If the category of all finitely generated (A, B)-projective A-modules is closed under taking A-syzygies, then $\text{fin.dim}(B) \leq \text{fin.dim}(A) + \text{fin.dim}({}_BA) + 3$, where $\text{fin.dim}(A)$ denotes the finitistic dimension of A, and where $\text{fin.dim}({}_BA)$ stands for the supremum of the projective dimensions of those direct summands of ${}_BA$ that have finite projective dimension. (2) If the extension $B \subseteq A$ is n-hereditary for a non-negative integer n, then $\text{gl.dim}(A) \leq \text{gl.dim}(B)+n$. Moreover, we show that the finitistic dimension of the trivially twisted extension of two algebras of finite finitistic dimension is again finite. Also, a new formulation of the finitistic dimension conjecture in terms of relative homological dimension is given. Our approach in this paper is completely different from the one in our earlier papers.

Keywords: Finitistic dimension; global dimension; relatively projective module; relatively hereditary extension.

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1. Introduction

In the representation theory of Artin algebras, there is a well-known conjecture: *For any Artin algebra, its finitistic dimension is finite.* This is the so-called finitistic dimension conjecture (see [3; 2, p. 409]). It is over 45 years old and remains open to date. The significance of this conjecture lies on the well-known fact that an affirmative answer to the finitistic dimension conjecture implies the validity of the other seven homological conjectures in the modern representation theory of Artin algebras (see [2, p. 409; 27]). To understand the conjecture, a new idea was introduced in [23, 24] to control finitistic dimension by using a chain of algebras with certain properties on their radicals. This is applicable for general finite-dimensional algebras. It turns out that, for a field $\mathbb{K}$, the following two statements are equivalent:

(1) For any finite-dimensional $\mathbb{K}$-algebra A, the finitistic dimension of A is finite.
(2) If C is a subalgebra of a finite-dimensional $\mathbb{K}$-algebra B such that the radical of C is a left ideal in B, and if B has finite finitistic dimension, then C has finite finitistic dimension.

This suggests that it would be interesting to consider a pair $B \subseteq A$ of algebras A and B, and to try bounding the finitistic dimension of the smaller algebra B by that of the bigger algebra A. In other words, for which extensions $B \subseteq A$ does the statement (2) hold? Such a consideration seems to be reasonable because the module category of A is sometimes much simpler than that of B. In fact, some of the discussions in this direction have been done already in [23, 24], where the representation-finite type and finite global dimension are involved for comparison of finitistic dimensions of B and A.

Recall that an Artin algebra A is called a *separable extension* of a subalgebra B of A with the same identity if the multiplication map $A \otimes_B A \to A$ splits as A–A-bimodules. This is a generalization of the notion of a separable algebra over a field. An extension $B \subseteq A$ of Artin algebras is called (left) *semisimple* if every left A-module is (A, B)-projective in the sense of Hochschild [15], or equivalently, the multiplication map $\mu : {}_AA \otimes_B X \to {}_AX$ of A-modules splits for every left A-module ${}_AX$, that is, there is a homomorphism $\varphi : X \to A \otimes_B X$ of A-modules such that $\varphi\mu$ is the identity map on X (for other equivalent conditions, see [2, Proposition 3.6, p. 202] or [13], for instance). Of course, one may define the right semisimple extension analogously by using right A-modules. The name "semisimple extension" is justified by the fact that a finite-dimensional $\mathbb{K}$-algebra over a field $\mathbb{K}$ is semisimple if and only if the extension $\mathbb{K} \subseteq A$ is semisimple. Clearly, a separable extension is both semisimple and right semisimple. In general, a semisimple extension does not have to be separable because a semisimple algebra over a field does not have to be a separable algebra. Another example of a semisimple extension is the extension $B \subseteq A$ such that B and A have the same radical. One may also construct new examples of semisimple extensions by using a result in [11]. A semisimple extension will also be called a 0-*hereditary* extension in this paper.

Now, let us introduce the notion of n-hereditary extensions for $1 \leq n < \infty$. An extension $B \subseteq A$ of algebras is called 1-*hereditary* (or hereditary) if every A-submodule of an (A, B)-projective A-module is (A, B)-projective. Thus, semisimple extensions are 1-hereditary extensions. The converse in general is not true. Again, the name "1-hereditary extension" is justified by the fact that a finite-dimensional $\mathbb{K}$-algebra A over a field $\mathbb{K}$ is hereditary if and only if the extension $\mathbb{K} \subseteq A$ is a 1-hereditary extension. Similarly, if the kernel of any homomorphism between two (A, B)-projective modules is (A, B)-projective, then we say that the extension $B \subseteq A$ is 2-*hereditary*. In general, an extension $B \subseteq A$ is said to be n-*hereditary* if, for any exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0$ of A-modules with X_j being (A, B)-projective for $0 \leq j \leq n-1$, the module X_n is (A, B)-projective. Clearly, an n-hereditary extension is an $(n+1)$-hereditary extension for $0 \leq n < \infty$. In this note, an extension $B \subseteq A$ of Artin algebras is said to be *relatively hereditary* if the extension $B \subseteq A$ is n-hereditary for some non-negative integer n.

Suppose that we have an extension $C \subseteq B$ of Artin algebras such that the radical of C, denoted by $\mathrm{rad}(C)$, is a left ideal of B. In this note, we shall compare the global and finitistic dimensions of C with that of B. In particular, we shall show that the statement (2) is true if the category of (B, C)-projective B-modules is closed under taking B-syzygies. This includes the case of n-hereditary extensions. More precisely, we shall prove the following general result.

Theorem 1.1. *Let A be an Artin algebra and B be a subalgebra of A such that the radical of B is a left ideal in A.*

(1) *Suppose that the category of all finitely generated (A, B)-projective A-modules is closed under taking A-syzygies. Then* $\mathrm{fin.dim}(B) \leq \mathrm{fin.dim}(A) + \mathrm{fin.dim}({}_BA) + 3$, *where* $\mathrm{fin.dim}(A)$ *denotes the finitistic dimension of A, and where* $\mathrm{fin.dim}({}_BA) = \max\{\mathrm{proj.dim}({}_BX) \mid X$ is a direct summand of ${}_BA$ *with* $\mathrm{proj.dim}({}_BX) < \infty\}$.
(2) *Suppose that A is an n-hereditary extension of B for a non-negative integer n. Then* $\mathrm{gl.dim}(A) \leq \mathrm{gl.dim}(B) + n \leq \mathrm{gl.dim}(A) + \mathrm{proj.dim}({}_BA) + n + 2$.

In general, for finitistic dimension, we cannot get $\mathrm{fin.dim}(A) \leq \mathrm{fin.dim}(B) + n$ if A is an n-hereditary extension of B. Thus, Theorem 1.1 shows that the notions of global and finitistic dimensions behave completely differently, even though the finitistic dimension of an algebra coincides with its global dimension if the global dimension is finite.

Note that, for semisimple extensions $B \subseteq A$, there were some discussions on their global, weakly global and finitistic dimensions (see, for example, [6, 8]). Here, the extensions we consider are more general, including semisimple extensions. Furthermore, Theorem 1.1 requires a radical condition which is motivated by considering finitistic dimension conjecture, and is not of homological feature, while results in the literature put homological conditions on the B-module A, for instance, ${}_BA$ or A_B is projective as a B-module. For details, we refer the reader to [6, 8].

As a direct corollary of Theorem 1.1, we have the following corollary.

Corollary 1.2. *Suppose that A is an Artin algebra and B is a subalgebra of A such that B and A have the same radical. If* $\text{fin.dim}(A) < \infty$, *then* $\text{fin.dim}(B) < \infty$.

As a consequence of Corollary 1.2, we have the following result which states that the gluing idempotent procedure preserves the finiteness of finitistic dimension (see [23]), and therefore generalizes the result [24, Corollary 3.11].

Corollary 1.3. *Let A_0, A_1 and A_2 be three algebras with A_0 semi-simple. Given surjective homomorphisms $f_i : A_i \to A_0$ of algebras for $i = 1, 2$, we denote by A the pullback of f_1 and f_2 over A_0. If the finitistic dimension of A_i is finite for $i = 1, 2$, then the finitistic dimension of A is finite.*

Let us remark that, for an extension $B \subseteq A$, even under the condition $\text{rad}(B) = \text{rad}(A)$, the module category of B can be much more complicated than that of A. An easy example illustrates this point. Let A be the algebra of 4×4 upper triangular matrices over a field $\mathbb{K}$, and let B be the subalgebra of A generated by the identity matrix and the radical of A. Then $\text{rad}(B) = \text{rad}(A)$, and B is representation-wild, while A is representation-finite with 10 non-isomorphic indecomposable modules. In fact, any algebra A with at least three arrows in its quiver contains a subalgebra B of wild type with $\text{rad}(B)$ equal to $\text{rad}(A)$.

Also, the proof of Theorem 1.1 implies the following result in which we only assume that A, viewed only as a right B-module, satisfies some homological condition, while in the literature such conditions are imposed on both sides.

Corollary 1.4. *Let A be an Artin algebra and B be a subalgebra of A such that* $\text{rad}(B)$ *is a left ideal in A and that the projective dimension of the right B-module A_B is finite. Then* $\text{fin.dim}(B) \leq \text{fin.dim}(A) + \text{proj.dim}(A_B) + 2$.

Finally, we give lower and upper bounds for the finitistic dimensions of algebras in terms of those of certain subalgebras. This shows, in particular, that the finiteness of finitistic dimensions is preserved by taking trivially twisted extensions, or dual extensions. For unexplained notions in the following result, we refer the reader to Sec. 3.

Theorem 1.5. *Suppose that A decomposes into a twisted tensor product $B \wedge C$ of its subalgebras B and C over a common maximal semisimple subalgebra of A. If* $\text{fin.dim}(B) = m < \infty$ *and* $\text{fin.dim}(C) = n < \infty$, *then* $m \leq \text{fin.dim}(A) \leq m+n < \infty$.

Note that Corollary 1.2 provides a partial answer to Question 1 in [24]; while Theorem 1.5 extends the result [23, Corollary 3.9].

Our approach in this paper is different from the earlier papers [23, 24, 26], namely, instead of employing the Igusa–Todorov function from [16], we only use properties of relatively projective modules to get our results. The proofs of

Theorems 1.1 and 1.5 will be given in Secs. 2 and 3, respectively, where we establish actually a relationship between the finitistic dimensions of the two algebras A and B under our assumptions. A variation of Theorem 1.1(1) can be found in Proposition 2.10. In Sec. 2, we also provide an equivalent formulation of the finitistic dimension conjecture in terms of extensions and relative global dimensions.

2. Proof of Theorem 1.1

In this section, we give a proof of the main result, Theorem 1.1. First, let us recall some definitions and introduce some notation.

Given an Artin R-algebra A over a commutative Artin ring R with identity, we consider the category A-mod of all finitely generated left A-modules. The usual duality of Artin algebra, defined by the injective hull of the R-module $R/\mathrm{rad}(R)$, is denoted by D. The nth syzygy operator of A-mod is denoted by Ω_A^n. For an A-module M, we use $\mathrm{rad}({}_AM)$ to denote the Jacobson radical of M, and $\mathrm{top}_A(M)$ to denote the top of M, that is, $\mathrm{top}_A(M) = M/\mathrm{rad}({}_AM)$; the projective dimension of M is denoted by $\mathrm{proj.dim}({}_AM)$.

For convenience, homomorphisms will be written on the opposite side of the scalars. Thus the composition of two homomorphisms $f : X \to Y$ and $g : Y \to Z$ of A-modules is written as fg which is a homomorphism from X to Z. In this way, $\mathrm{Hom}_A(X, Y)$ becomes naturally an $\mathrm{End}_A(X)$–$\mathrm{End}_A(Y)$-bimodule.

If B is a subalgebra of an Artin algebra A with the same identity, we say that $B \subseteq A$ is an *extension*. A subcategory $\mathcal{C}$ of A-mod is said to be *closed under syzygies* if, for any module X in $\mathcal{C}$, the first syzygy $\Omega_A(X)$ of X belongs to $\mathcal{C}$. Thus, if $\mathcal{C}$ is closed under syzygies, then $\Omega_A^i(X) \in \mathcal{C}$ for all $i \geq 1$ whenever $X \in \mathcal{C}$.

By definition, the *finitistic dimension of an A-module* ${}_AM$, denoted by $\mathrm{fin.dim}({}_AM)$, is

$$\begin{aligned}&\mathrm{fin.dim}({}_AM)\\ &\quad = \sup\{\mathrm{proj.dim}({}_AM') \mid M' \text{ is a direct summand of } M,\ \mathrm{proj.dim}({}_AM') < \infty\},\end{aligned}$$

and the *finitistic dimension* of the algebra A, denoted by $\mathrm{fin.dim}(A)$, is

$$\mathrm{fin.dim}(A) = \sup\{\mathrm{fin.dim}({}_AM) \mid M \in A\text{-mod}\}.$$

Note that $\mathrm{fin.dim}({}_AM)$ is always finite if ${}_AM \in A$-mod.

Similarly, one may define the right finitistic dimension of A by using the projective dimensions of right A-modules. In general, $\mathrm{fin.dim}(A) \neq \mathrm{fin.dim}(A^{\mathrm{op}})$, where A^{op} stands for the opposite algebra of A. However, if we use the injective dimensions of left A-modules to define the finitistic injective dimension of A, denoted by $\mathrm{fin.inj.dim}(A)$, then $\mathrm{fin.dim}(A) = \mathrm{fin.inj.dim}(A^{\mathrm{op}})$.

The famous finitistic dimension conjecture states that there exists a uniform bound for the finite projective dimensions of all finitely generated (left) A-modules of finite projective dimension, namely $\mathrm{fin.dim}(A) < \infty$. This conjecture is closely related to the Nakayama conjecture, Gorenstein symmetry conjecture, Wakamatsu tilting conjecture, and other homological conjectures (for details, see [2, 4, 24, 27]).

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. *Let A be an Artin algebra, and let M be an A-module.*

(1) *If there is an exact sequence*

$$0 \to X_s \to \cdots \to X_1 \to X_0 \to M \to 0$$

of A-modules with $\mathrm{proj.dim}({}_AX_i) \leq k$ *for all i, then* $\mathrm{proj.dim}({}_AM) \leq s + k$.

(2) (*Nakayama lemma*): *If* $\mathrm{rad}({}_AM) = M$, *then* $M = 0$.

(4) *If I is an ideal in A, then, for every A/I-module X,* $\mathrm{Hom}_A(M, X) \simeq \mathrm{Hom}_A(M/IM, X)$. *In particular, if X is a semisimple A-module, then* $\mathrm{Hom}_A(M, X) \simeq \mathrm{Hom}_A(M/\mathrm{rad}(M), X)$.

The following lemma establishes a way of lifting modules over a subalgebra to modules over a given algebra.

Lemma 2.2 ([23, Erratum, Lemma 0.1, p. 325]). *Let A and B be two Artin algebras with B a subalgebra of A. Suppose that* $\mathrm{rad}(B)$ *is a left ideal of A. If X is a B-module, then $\Omega^i_B(X)$ is an A-module for all $i \geq 2$.*

Suppose that $B \subseteq A$ is an extension of Artin algebras. If X is an A-module, then X can be considered as a B-module by restriction. If $\mathrm{rad}(B) \subseteq \mathrm{rad}(A)$, we have that $\mathrm{top}_B(X) \simeq {}_B\mathrm{top}_A(X) \oplus \mathrm{rad}({}_AX)/\mathrm{rad}({}_BX)$. If $\mathrm{rad}(B)$ is a left ideal in A, then $\mathrm{rad}({}_BX)$ is an A-module for any A-module X. Moreover, since $\mathrm{rad}({}_BP)$ is an A-module for every projective B-module P, we see that $\Omega_B(X)$ is an A-module for any A-module X.

We should note that, for the extension $B \subseteq A$, the condition that $\mathrm{rad}(B)$ is a left ideal in A does not imply that $\mathrm{rad}({}_BY)$ has an A-module structure for each B-module Y. This can be seen by an example in [23, Erratum].

If Y is a B-module, then the map $f_Y : Y \to {}_BA \otimes_B Y$ given by $y \mapsto 1 \otimes y$ for $y \in Y$ is a homomorphism of B-modules. Note that if ${}_AX$ is an A-module, then the multiplication map $\mu_X : A \otimes_B X \to X$ is a homomorphism of A-modules. Thus μ_X is also a homomorphism of B-modules by restriction. Clearly, we have $f_X\mu_X = \mathrm{id}_X$. This shows the following lemma.

Lemma 2.3. *Let A be an Artin algebra and B be a subalgebra of A. Then, for any A-module X, there is a split exact sequence of B-modules*:

$$0 \to {}_BX \to {}_BA \otimes_B X \to (A/B) \otimes_B X \to 0.$$

Note that A can be regarded as a B–B-bimodule by restriction. Thus the quotient A/B of the B–B-bimodule A by its subbimodule B has a natural B–B-bimodule structure.

Recall that, for an extension $B \subseteq A$ of algebras, an A-module X is called (A, B)-*projective* if the multiplication map $\mu_X : A \otimes_B X \to X$ splits as A-modules. This is equivalent to saying that ${}_AX$ is a direct summand of ${}_AA \otimes_B X$. Clearly, each projective A-module Y is (A, B)-projective, and every A-module of the form $A \otimes_B Y$ with $Y \in B$-mod is (A, B)-projective. The full subcategory of A-mod consisting of all (A, B)-projective A-modules will be denoted by $\mathscr{P}(A, B)$. Note that $\mathscr{P}(A, B)$ is contravariantly finite in A-mod (see [15, Proposition 2]). Moreover, it is functorially finite in A-mod (see [17]).

Similar to the usual definitions of projective dimension and global dimension, one can employ (A, B)-projective modules to define the so-called relative projective dimension of an A-module and the relative global dimension of A with respect to B, respectively. That is, for an A-module X, we define the *relative projective dimension* of X to be the minimal number n such that there is an exact sequence

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to X \to 0$$

with the following two properties:

(1) All P_j are (A, B)-projective.
(2) For any (A, B)-projective module X', the sequence

$$0 \to \mathrm{Hom}_A(X', P_n) \to \cdots \to \mathrm{Hom}_A(X', P_1)$$
$$\to \mathrm{Hom}_A(X', P_0) \to \mathrm{Hom}_A(X', X) \to 0$$

is exact.

If such an exact sequence does not exist, then we say that the relative projective dimension of X is infinite. The *relative global dimension* of the extension $B \subseteq A$ is the supremum of the relative projective dimensions of A-modules, denoted by $\mathrm{gl.dim}(A, B)$.

Also, the relative derived functors Tor and Ext can be defined. For details, we refer the reader to [5, 15]. We define the *relative finitistic dimension* of the extension $B \subseteq A$ to be the supremum of the relative projective dimensions of those A-modules with finite relative projective dimension, and denote it by $\mathrm{fin.dim}(A, B)$. Note that an extension $B \subseteq A$ is semisimple if and only if $\mathrm{gl.dim}(A, B) = 0$ if and only if every A-module is (A, B)-projective. Clearly, if the extension $B \subseteq A$ is n-hereditary, then $\mathrm{gl.dim}(A, B) \leq n$ because the multiplication map $A \otimes_B X \to X$ is a right $\mathscr{P}(A, B)$-approximation of X for any A-module X.

The following lemma describes (A, B)-projective modules. The first statement is due to Hochschild [15], and the second follows from the fact that the zero A-module is (A, B)-projective.

Lemma 2.4. *Let $B \subseteq A$ be an extension of Artin algebras.*

(1) *If ${}_BX$ is a B-module, then $A \otimes_B X$ is (A, B)-projective.*

(2) *If the given extension $B \subseteq A$ is relatively hereditary, then $\mathscr{P}(A, B)$ is closed under kernels of surjective homomorphisms in $\mathscr{P}(A, B)$. In particular, it is closed under taking A-syzygies.*

The next lemma establishes a relationship between different syzygies.

Lemma 2.5. *Let A be an Artin algebra and B be a subalgebra of A such that* $\operatorname{rad}(B)$ *is a left ideal of A. Then, for any A-module Y, we have an isomorphism ${}_A\Omega_B(Y) \simeq {}_A\Omega_A(A \otimes_B Y)$ as A-modules.*

Proof. Let $h : P \to Y$ be a projective cover of the B-module ${}_BY$. Since B is a subalgebra of A, we have an exact sequence $0 \to B \to A \to A/B \to 0$ of B–B-bimodules, where $(A/B)_B$ is semisimple because $\operatorname{rad}(B)$ is a left ideal in A and $(A/B)\operatorname{rad}(B) = 0$. Then we obtain the following exact and commutative diagram of B-modules:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_B(Y) \arrow[r] & P \arrow[r, "h"] \arrow[d, "f_P"'] & Y \arrow[r] \arrow[d, "f_Y"'] & 0 \\
 & & A \otimes_B P \arrow[r, "1\otimes h"] \arrow[d, "\pi_P"'] & A \otimes_B Y \arrow[r] \arrow[d, "\pi_Y"'] & 0 \\
 & & (A/B) \otimes_B P \arrow[r] \arrow[d] & (A/B) \otimes_B Y \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where the left column is exact since ${}_BP$ is projective, and where the right column is split exact by Lemma 2.3. Since A/B is semisimple as a right B-module, $D(A/B)$ is a semisimple left B-module. It follows from Lemma 2.1(3) that $D((A/B) \otimes_B P) \simeq \operatorname{Hom}_B(P, D(A/B)) \simeq \operatorname{Hom}_B(P/\operatorname{rad}({}_BP), D(A/B))$. Similarly, $D((A/B) \otimes_B Y) \simeq \operatorname{Hom}_B(Y/\operatorname{rad}({}_BY), D(A/B))$. Since h is a projective cover, we have $P/\operatorname{rad}({}_BP) \simeq Y/\operatorname{rad}({}_BY)$. Hence $D((A/B) \otimes_B P) \simeq D((A/B) \otimes_B Y)$. This implies that $(A/B) \otimes_B P \simeq (A/B) \otimes_B Y$ as B-modules. Since all modules considered have finite length, the surjectivity of the map in the last row of the diagram implies that it is also an isomorphism. Note that $A \otimes_B P$ is a projective A-module. So the kernel of $1 \otimes h$ is isomorphic to $\Omega_A(A \otimes_B Y) \oplus Q$ for some projective A-module Q. Thus we have the following isomorphism of B-modules:

$$(*) \quad {}_B\Omega_B(Y) \simeq {}_B\Omega_A(A \otimes_B Y) \oplus {}_BQ.$$

Next, we shall show that $(*)$ is even and an isomorphism of A-modules. In fact, since $\operatorname{rad}(B)$ is a nilpotent left ideal in A, we know $\operatorname{rad}(B) \subseteq \operatorname{rad}(A)$. This implies that there are injective homomorphisms of A-modules: $\mathrm{rad}(B)(A \otimes_B P) \to \mathrm{rad}(A)(A \otimes_B P)$ and $\mathrm{rad}(B)(A \otimes_B Y) \to \mathrm{rad}(A)(A \otimes_B Y)$ given by inclusions. Note that $f_P|_{\mathrm{rad}(B)P} : \mathrm{rad}(B)P \to \mathrm{rad}(B)(A \otimes_B P)$ and $f_Y|_{\mathrm{rad}(B)Y} : \mathrm{rad}(B)Y \to \mathrm{rad}(B)(A \otimes_B Y)$ are injective homomorphisms of A-modules. By [2, Theorem 2.2, p. 7], the A-homomorphism $1 \otimes h : A \otimes_B P \to A \otimes_B Y$ can be decomposed as $\binom{h_1}{0} : P_1 \oplus Q \to A \otimes_B Y$ such that $A \otimes_B P = P_1 \oplus Q$ with P_1 and Q projective A-modules, and that $h_1 = (1 \otimes h)|_{P_1} : P_1 \to A \otimes_B Y$ is a projective cover of the A-module $A \otimes_B Y$. Note that the A-module Q in this decomposition is the same as the Q appearing in $(*)$. Then we have an exact sequence of A-modules:

$$0 \to \Omega_A(A \otimes_B Y) \oplus \mathrm{rad}(A)Q \to \mathrm{rad}({}_A A \otimes_B P) \to \mathrm{rad}({}_A A \otimes_B Y) \to 0.$$

Let $h' = (1 \otimes h)|_{\mathrm{rad}({}_B A \otimes_B P)} : \mathrm{rad}({}_B A \otimes_B P) \to \mathrm{rad}({}_B A \otimes_B Y)$. Then we may form the following commutative diagram of A-modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_B(Y) \arrow[r] \arrow[d] & \mathrm{rad}({}_B P) \arrow[r] \arrow[d] & \mathrm{rad}({}_B Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(h') \arrow[r] \arrow[d] & \mathrm{rad}({}_B A \otimes_B P) \arrow[r, "h'"] \arrow[d] & \mathrm{rad}({}_B A \otimes_B Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_A(A \otimes_B Y) \oplus \mathrm{rad}(A)Q \arrow[r] & \mathrm{rad}({}_A P_1 \oplus {}_A Q) \arrow[r] & \mathrm{rad}({}_A A \otimes_B Y) \arrow[r] & 0
\end{tikzcd}$$

Since the two maps in the middle column are injective, we know that the composition of the two maps in the first column are injective, too. Thus we have an injective homomorphism

$$\Omega_B(Y) \to \Omega_A(A \otimes_B Y) \oplus \mathrm{rad}(A)Q$$

of A-modules, which can be composed further with the canonical inclusion $\Omega_A(A \otimes_B Y) \oplus \mathrm{rad}(A)Q \to \Omega_A(A \otimes_B Y) \oplus Q$. In this way, we get an injective homomorphism from the A-module $\Omega_B(Y)$ to the A-module $\Omega_A(A \otimes_B Y) \oplus Q$. As a result, we obtain an isomorphism:

$$\Omega_B(Y) \simeq \Omega_A(A \otimes_B Y) \oplus Q \simeq \Omega_A(A \otimes_B Y) \oplus \mathrm{rad}({}_A Q)$$

as A-modules by $(*)$. Thus $Q = 0$ by the Nakayama lemma (see Lemma 2.1(2)), and $\Omega_B(Y) \simeq \Omega_A(A \otimes_B Y)$ as A-modules. This shows also that $1 \otimes h$ is a projective cover of the A-module $A \otimes_B Y$. The proof is completed. $\square$

As an immediate consequence of Lemma 2.5, we have the following corollary in which we do not assume that B is a direct summand of ${}_B A_B$ as a B–B-bimodule (see [6, 7, Lemma 4.2, Theorem 4.3] for a comparison).

Corollary 2.6. *Let A be an Artin algebra and B be a subalgebra of A such that* $\mathrm{rad}(B)$ *is a left ideal in A and that* $\mathrm{proj.dim}(A_B) < \infty$. *Then* $\mathrm{fin.dim}(B) \leq \mathrm{fin.dim}(A) + \mathrm{proj.dim}(A_B) + 2$.

Proof. Suppose $\text{proj.dim}(A_B) = n < \infty$. Let ${}_BX$ be a B-module with $\text{proj.dim}({}_BX) < \infty$. Then $Y := \Omega_B^{n+2}(X)$ is an A-module by Lemma 2.2. If $0 \to P_m \to \cdots \to P_1 \to P_0 \to {}_BY \to 0$ is a minimal projective resolution of the B-module ${}_BY$, then the sequence $0 \to A \otimes_B P_m \to \cdots \to A \otimes_B P_1 \to A \otimes_B P_0 \to A \otimes_B Y \to 0$ is exact since $\text{Tor}_j^B(A_B, Y) = \text{Tor}_j^B(A, \Omega_B^{n+2}(X)) \simeq \text{Tor}_{n+j+2}^B(A_B, X) = 0$ for $j \geq 1$. Moreover, the proof of Lemma 2.5 shows that this sequence is also a minimal projective resolution of the A-module ${}_AA \otimes_B Y$. Thus $\text{proj.dim}({}_BY) = \text{proj.dim}({}_AA \otimes_B Y) \leq \text{fin.dim}(A)$. This implies that $\text{proj.dim}({}_BX) \leq \text{fin.dim}(A) + n + 2$ and $\text{fin.dim}(B) \leq \text{fin.dim}(A) + n + 2$. $\square$

Note that, by [26, Corollary 3.16], we may replace "$\text{proj.dim}(A_B) < \infty$" by "the Gorenstein-projective dimension of the right B-module A_B is finite" in Corollary 2.6 to get a more general result.

Another consequence of Lemma 2.5 is the following result.

Corollary 2.7. *Let A be an Artin algebra and B be a subalgebra of A such that* $\text{rad}(B)$ *is a left ideal of A.*

(1) *If ${}_AY$ is an (A, B)-projective A-module, then ${}_A\Omega_B(Y) \simeq \Omega_A(Y) \oplus \Omega_A(Y')$ as A-modules, where Y' is an A-module.*
(2) *Let $i \geq 2$ be an integer, and let X be a B-module. If $\Omega_B^i(X)$ is (A, B)-projective, then*

$$ {}_A\Omega_B^{i+1}(X) \simeq {}_A\Omega_A(A \otimes_B \Omega_B^i(X)) \simeq {}_A\Omega_A(\Omega_B^i(X)) \oplus {}_A\Omega_A(Q), $$

where Q is the A-module $\text{Ker}(\mu_{\Omega_B^i(X)})$.
(3) *If $\mathscr{P}(A, B)$ is closed under taking A-syzygies, then, for every A-module Y, the A-module ${}_A\Omega_B(Y)$ is (A, B)-projective. In particular, if the extension $B \subseteq A$ is relatively hereditary, then, for any A-module ${}_AY$, the A-module $\Omega_B(Y)$ is (A, B)-projective.*

Proof. (1) Since Y is (A, B)-projective, there is an A-module Y' such that $A \otimes_B Y \simeq Y \oplus Y'$ as A-modules. By Lemma 2.5, we have

$$ {}_A\Omega_B(Y) \simeq {}_A\Omega_A(A \otimes_B Y) \simeq \Omega_A(Y) \oplus \Omega_A(Y'). $$

(2) Since $\Omega_B^i(X)$ is an A-module for $i \geq 2$ by Lemma 2.2, it follows that (2) is a consequence of (1) by taking $Y = \Omega_B^i(X)$.
(3) By Lemma 2.5, $\Omega_B(Y)$ is isomorphic to $\Omega_A(A \otimes_B Y)$ as A-modules. Since the latter is (A, B)-projective by assumption and Lemma 2.4(1), we see that $\Omega_B(Y)$ is (A, B)-projective. The last statement in (3) follows then from Lemma 2.4(2) since $\mathscr{P}(A, B)$ being closed under kernels of surjective homomorphisms between

(A,B)-projective modules implies that $\mathscr{P}(A,B)$ is closed under taking A-syzygies. $\square$

As a consequence of Corollary 2.7(1), we have the following corollary.

Corollary 2.8. *Let A be an Artin algebra and B be a subalgebra of A such that* $\mathrm{rad}(B)$ *is a left ideal in A. Then, for any (A,B)-projective A-module Y with* $\mathrm{proj.dim}({}_BY) < \infty$, *we have* $\mathrm{proj.dim}({}_B\Omega_A(Y)) < \infty$.

Proof. The corollary follows immediately from Corollary 2.7(1) since ${}_B\Omega_B(Y) \simeq {}_B\Omega_A(Y) \oplus {}_B\Omega_A(Y')$ as B-modules, where Y' is an A-module. $\square$

Lemma 2.9. *Suppose that $B \subseteq A$ is an extension of Artin algebras such that* $\mathrm{rad}(B)$ *is a left ideal in A.*

(1) *Let Y be an A-module and $n \geq 1$ be an integer or infinity. If ${}_A\Omega_B^i(Y)$ is (A,B)-projective for $0 \leq i \leq n-1$, then we have an isomorphism of A-modules*:

$$ {}_A\Omega_B^j(Y) \simeq {}_A\Omega_A^j(Y) \oplus \bigoplus_{i=1}^{j} {}_A\Omega_A^{j-i+1}(T_i) $$

for $1 \leq j \leq n$, where T_i is the A-module $\mathrm{Ker}(\mu_{\Omega_B^i(Y)})$.

(2) *Let ${}_BX$ be a B-module. If there is an integer $n \geq 2$ such that ${}_A\Omega_B^j(X)$ is (A,B)-projective for all $j \geq n$, then there is an isomorphism of A-modules*:

$$ {}_A\Omega_B^{j+n}(X) \simeq {}_A\Omega_A^j(\Omega_B^n(X)) \oplus \bigoplus_{i=1}^{j} {}_A\Omega_A^{j-i+1}(T_i) $$

for all $j \geq 1$, where T_i are some A-modules.

Proof. Note that (2) follows from (1) if we put $Y = \Omega_B^n(X)$. So we need only to prove (1). If $1 \leq j \leq n$, then ${}_A\Omega_B^{j-1}(Y)$ is (A,B)-projective by assumption. Thus we have

$$ \begin{aligned} \Omega_B^j(Y) &= \Omega_B(\Omega_B^{j-1}(Y)) \\ &\simeq \Omega_A(\Omega_B^{j-1}(Y)) \oplus \Omega_A(Q_j) \quad \text{(Corollary 2.7(1))} \\ &\simeq \Omega_A\big(\Omega_A(\Omega_B^{j-2}(Y)) \oplus \Omega_A(Q_{j-1})\big) \oplus \Omega_A(Q_j) \\ &= \Omega_A^2(\Omega_B^{j-2}(Y)) \oplus \Omega_A^2(Q_{j-1}) \oplus \Omega_A(Q_j) \\ &\;\vdots \\ &\simeq \Omega_A^j(Y) \oplus \bigoplus_{i=1}^{j} {}_A\Omega_A^{j-i+1}(Q_i), \end{aligned} $$

where Q_i are some A-modules. Thus (1) follows. $\square$

Proof of Theorem 1.1. We first prove Theorem 1.1(1). Let s be the finitistic dimension of A. Suppose that ${}_BX$ is a B-module such that $\mathrm{proj.dim}({}_BX) = m < \infty$. We may assume that $m \geq 3$. Then $Y' := \Omega_B^2(X)$ is an A-module by Lemma 2.2. Since we assume that $\mathscr{P}(A, B)$ is closed under taking syzygies, we infer that $\Omega_B^j(Y')$ is (A, B)-projective for all $j \geq 1$ by Corollary 2.7(3). Now we set $Y := \Omega_B(Y') = \Omega_B^3(X)$. Then ${}_A\Omega_B^j(Y)$ is (A, B)-projective for all $j \geq 0$. By Lemma 2.9(1), we have

$$0 = {}_A\Omega_B^{m+1}(X) \simeq {}_A\Omega_A^{m-2}(Y) \oplus \bigoplus_{i=1}^{m-2} {}_A\Omega_A^{m-2-i+1}(Q_i)$$

with Q_i certain A-modules. Thus $\mathrm{proj.dim}({}_AY) \leq m - 3$. Let

$$0 \to P_t \to \cdots \to P_1 \to P_0 \to Y \to 0$$

be a minimal projective resolution of the A-module Y with $t \leq s$. Since $\mathscr{P}(A, B)$ is closed under taking syzygies, we see that $\Omega_A^i(Y)$ is (A, B)-projective for all $i \geq 0$. Then the projective dimension of the B-module ${}_B\Omega_A^i(Y)$ is finite by Corollary 2.8, and the restriction to B of the projective A-module P_i in the sequence has finite projective dimension for all i. Thus, by Lemma 2.1, we see that $\mathrm{proj.dim}({}_BY) \leq t + \mathrm{fin.dim}({}_BA)$, where $\mathrm{fin.dim}({}_BM)$ is the supremum of projective dimensions of those B-modules M' which are direct summands of ${}_BM$ with $\mathrm{proj.dim}({}_BM') < \infty$. Hence $\mathrm{proj.dim}({}_BX) \leq \mathrm{fin.dim}({}_BA) + s + 3$. This means that $\mathrm{fin.dim}(B) \leq \mathrm{fin.dim}(A) + \mathrm{fin.dim}({}_BA) + 3$. Note that we do not need the finiteness of $\mathrm{fin.dim}(A)$ in the above arguments.

Now we turn to the proof of Theorem 1.1(2). Let us show the first inequality. If $\mathrm{gl.dim}(B)$ is infinite, then Theorem 1.1(2) is trivially true. So we assume that $\mathrm{gl.dim}(B) = m < \infty$. Let Y be an A-module. Then the module $\Omega_A^i(Y)$ is (A, B)-projective for $i \geq n$ since it is the kernel of a morphism f in a long exact sequence of length n of projective A-modules. Thus, by Lemma 2.9(1), we have the following isomorphism of A-modules for $s \geq 1$:

$${}_A\Omega_B^s(\Omega_A^n(Y)) \simeq {}_A\Omega_A^s(\Omega_A^n(Y)) \oplus \bigoplus_{i=1}^{s} {}_A\Omega_A^{s-i+1}(T_i),$$

with T_i an A-module for all i. This shows that $\mathrm{proj.dim}({}_A\Omega_A^n(Y)) \leq m$ and $\mathrm{proj.dim}({}_AY) \leq m + n$. Thus $\mathrm{gl.dim}(A) \leq \mathrm{gl.dim}(B) + n$.

It remains to show the second inequality in Theorem 1.1(2). Suppose that X is a B-module. Then $\Omega_B^2(X)$ is an A-module by Lemma 2.2. Using a change of ring theorem for the inclusion map $B \hookrightarrow A$ (see, for example, [20, Theorem 4.3.1, p. 99]), we infer that $\mathrm{proj.dim}({}_B\Omega_B^2(X)) \leq \mathrm{proj.dim}({}_A\Omega_B^2(X)) + \mathrm{proj.dim}({}_BA) \leq \mathrm{gl.dim}(A) + \mathrm{proj.dim}({}_BA)$. This implies that $\mathrm{proj.dim}({}_BX) \leq \mathrm{gl.dim}(A) + \mathrm{proj.dim}({}_BA) + 2$. Thus $\mathrm{gl.dim}(B) \leq \mathrm{gl.dim}(A) + \mathrm{proj.dim}({}_BA) + 2$. This finishes the proof of Theorem 1.1. $\square$

The following result, which is a variation of Theorem 1.1(1), is implied by the proof of Theorem 1.1(1).

Proposition 2.10. *Let B be a subalgebra of an Artin algebra A such that* $\mathrm{rad}(B)$ *is a left ideal in A. Suppose that there is an integer $s \geq 2$ such that $\Omega_B^j(X)$ is (A, B)-projective for all $j \geq s$ and all B-modules X with* $\mathrm{proj.dim}({}_BX) < \infty$. *Then* $\mathrm{fin.dim}(B) \leq \mathrm{fin.dim}(A) + \mathrm{fin.dim}({}_BA) + s$.

Proof. Let X be a B-module with $\mathrm{proj.dim}({}_BX) < \infty$. Put $Y := \Omega_B^s(X)$. Then $\Omega_B^j(Y)$ is an A-module for all $j \geq 0$. Since $\Omega_B^j(Y) = \Omega_B^{j+s}(X)$ is (A, B)-projective for all $j \geq 0$ by assumption, we know that ${}_A\Omega_A^j(Y)$ is (A, B)-projective for all $j \geq 1$ by Lemma 2.9(1). Now the proof of Theorem 1.1 works smoothly. So we finally obtain that $\mathrm{proj.dim}({}_BX) \leq \mathrm{fin.dim}(A) + \mathrm{fin.dim}({}_BA) + s$ and that $\mathrm{fin.dim}(B) \leq \mathrm{fin.dim}(A) + \mathrm{fin.dim}({}_BA) + s$. $\square$

Before we start with the proof of Corollary 1.2, let us mention some properties of semisimple extensions, which may help to understand arguments in our later proofs, though they are not used in the proofs.

Separable and semisimple extensions have been studied for a long time in both commutative algebra and non-commutative algebra. The next lemma collects some simple but interesting properties of semisimple extensions. For further information, we refer the reader to [8, 13, 15].

Lemma 2.11. (1) *Every separable extension is semisimple.*

(2) *If an extension $B \subseteq A$ of Artin algebras is semisimple, then, for any A-module X, we have ${}_AA \otimes_B X \simeq {}_AX \oplus \mathrm{Ker}(\mu_X)$, where μ_X is the multiplication map from $A \otimes_B X$ to X.*

(3) *Let $C \subseteq B \subseteq A$ be extensions of Artin algebras.*

  (a) *If the extension $C \subseteq A$ is semisimple, then $B \subseteq A$ is semisimple.*

  (b) *If the extensions $C \subseteq B$ and $B \subseteq A$ are semisimple, then $C \subseteq A$ is semisimple.*

Proof of Corollary 1.2. We need the following lemma which shows that one can get a semisimple extension $B \subseteq A$ if $\mathrm{rad}(B) = \mathrm{rad}(A)$.

Lemma 2.12 ([9]). *Let A be an Artin R-algebra and B be a subalgebra of A with* $\mathrm{rad}(B)$ *equal to* $\mathrm{rad}(A)$. *Then, for any A-module X, the exact sequence*

$$0 \to \mathrm{Ker}(\mu_X) \to {}_AA \otimes_B X \xrightarrow{\mu_X} {}_AX \to 0$$

of A-modules splits, and the A-module $\mathrm{Ker}(\mu_X)$ *is semisimple, where μ_X is the multiplication map.*

Proof. For the convenience of the reader, we include here a proof which is shorter than the one in [9]. By ℓ_R we denote the R-length of modules. Let $K_X = \mathrm{Ker}(\mu_X)$.

Then we may form the following exact commutative diagram in A-mod:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_X & \longrightarrow & {}_AA\otimes_B X & \longrightarrow & {}_AX & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow & & \downarrow & & \\ & & \mathrm{top}_A(K_X) & \xrightarrow{\alpha} & \mathrm{top}_A({}_AA\otimes_B X) & \longrightarrow & \mathrm{top}_A(X) & \longrightarrow & 0. \end{array}$$

Note that ${}_BK_X \simeq {}_B(A/B)\otimes_B X$ as B-modules by Lemma 2.3 and that the B-module $(A/B)\otimes_B X$ is semisimple. Thus ${}_BK_X = \mathrm{top}_B(K_X) = {}_B\mathrm{top}_A(K_X)$. Here we use the fact that ${}_B\mathrm{top}_A(X) = \mathrm{top}_B(X)$ if $\mathrm{rad}(A) = \mathrm{rad}(B)A$ (see [24, Lemma 3.6]). It follows that $\ell_R(K_X) = \ell_R(\mathrm{top}_A(K_X))$ and K_X is a semisimple A-module. Thus β is an isomorphism. Now we claim that α is injective. In fact, the upper row in the above diagram splits as B-modules. This means that $\mathrm{top}_B(A\otimes_B X) \simeq \mathrm{top}_B(X) \oplus \mathrm{top}_B(K_X)$. Thus ${}_B\mathrm{top}_A(A\otimes_B X) \simeq {}_B\mathrm{top}_A(Y) \oplus {}_B\mathrm{top}_A(K_X)$. This implies that $\ell_R(\mathrm{top}_A(A\otimes_B X)) = \ell_R(\mathrm{top}_A(Y)) + \ell_R(\mathrm{top}_A(K_X))$. Hence α is injective. This yields that, as an induced sequence of a split exact sequence, the upper row in the above commutative diagram splits. The proof is completed. $\square$

Thus Corollary 1.2 follows immediately from Theorem 1.1 together with Lemma 2.12. $\square$

In the following, we consider some applications of our main result. As the first application of Corollary 1.2, we consider the pullback of two algebras of finite finitistic dimension, and give a proof of Corollary 1.3.

Proof of Corollary 1.3. By definition, $A = \{(x_1, x_2) \in A_1 \oplus A_2 \mid f_1(x_1) = f_2(x_2)\}$. The radical of $A_1 \oplus A_2$ is $\mathrm{rad}(A_1)\oplus$ $\mathrm{rad}(A_2)$. Since A_0 is semisimple, $\mathrm{rad}(A_i)$ is mapped to zero under f_i. This implies that $\mathrm{rad}(A_1)\oplus$ $\mathrm{rad}(A_2) \subseteq \mathrm{rad}(A)$. The pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{p_1} & A_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow[f_2]{} & A_0 \end{array}$$

shows that the projection p_i is surjective since each f_i is surjective. Thus $\mathrm{rad}(A)$ is mapped to $\mathrm{rad}(A_i)$ under p_i. This yields that $\mathrm{rad}(A)$ is included in $\mathrm{rad}(A_1) \oplus \mathrm{rad}(A_2)$, and thus $\mathrm{rad}(A) = \mathrm{rad}(A_1) \oplus \mathrm{rad}(A_2)$. Now Corollary 1.3 follows from Corollary 1.2. $\square$

The next corollary is a consequence of Theorem 1.1.

Corollary 2.13. *Let B be a subalgebra of an Artin algebra A such that* $\mathrm{rad}(B)$ *is a left ideal in A.*

(1) *Suppose that $\mathscr{P}(A,B)$ is closed under taking A-syzygies. If the global dimension of A is finite, then the finitistic dimension of B is finite.*
(2) *Suppose that the extension $B \subseteq A$ is semisimple. If I is an ideal in A such that* $\mathrm{fin.dim}(A/I)$ *is finite, then* $\mathrm{fin.dim}(B/(B \cap I))$ *is finite.*

Proof. (1) This follows from Theorem 1.1(1), because $\mathrm{gl.dim}(A) = \mathrm{fin.dim}(A)$ if $\mathrm{gl.dim}(A)$ is finite. To prove (2), we shall prove the following statement:

(#) If $B \subseteq A$ is a semisimple extension and I is an ideal in A, then the extension $\overline{B} \subseteq \overline{A}$ is semisimple, where $\overline{A} = A/I$ and $\overline{B}$ denotes the image of B under the canonical map from A to A/I.

In fact, the inclusion map from B to A induces an inclusion map from $\overline{B}$ to $\overline{A}$. Note that $\mathrm{rad}(\overline{B})$ is a left ideal in $\overline{A}$. Let X be an $\overline{A}$-module. We may regard X as an A-module via the canonical map. Thus the map $A \otimes_B X \to {}_AX$ is a split map. Now we apply $\overline{A} \otimes_A -$ to this split map and get a split homomorphism $\overline{A} \otimes_B X \to {}_{\overline{A}}X$. Here we used the fact that $\overline{A} \otimes_A X = {}_{\overline{A}}X$. Since X is an $\overline{A}$-module with $(I \cap B)X = 0$, we have $\overline{A} \otimes_B X = \overline{A} \otimes_{\overline{B}} X$. Thus the multiplication map $\overline{A} \otimes_{\overline{B}} X \to {}_{\overline{A}}X$ splits. This proves the statement (#). Thus (2) follows from Theorem 1.1. □

Before we deduce further consequences of Theorem 1.1, let us make a few remarks on semisimple extensions.

Remark. The statement (#) shows that an extension $B \subseteq A$ with $\mathrm{rad}(B)$ an *ideal* in A is semisimple if and only if the extension is radical-equal, that is, $\mathrm{rad}(B) = \mathrm{rad}(A)$. In fact, if $I := \mathrm{rad}(B)$ is an ideal in A and the extension is semisimple, then the induced extension $\overline{B} \subseteq \overline{A}$ is semisimple. Since $\overline{B}$ is a semisimple algebra, every $\overline{B}$-module is projective. This implies that every $\overline{A}$-module is projective. Hence $\overline{A}$ is semisimple and $\mathrm{rad}(B) = \mathrm{rad}(A)$. Similarly, if an extension $B \subseteq A$ with $\mathrm{rad}(B)$ a *left ideal* in A is semisimple, then $\mathrm{rad}(B)A = \mathrm{rad}(A)$, that is, the inclusion map is radical-full (see [24]). In general, if an extension $B \subseteq A$ of Artin algebras with $\mathrm{rad}(B)$ contained in $\mathrm{rad}(A)$ is semisimple, then $A\mathrm{rad}(B)A = \mathrm{rad}(A)$.

Since monomial algebras have finite finitistic dimension [12], we have the following result.

Corollary 2.14. *Let B be a subalgebra of a finite-dimensional monomial $\mathbb{K}$-algebra A over a field $\mathbb{K}$ such that* $\mathrm{rad}(B)$ *is a left ideal in A. If $\mathscr{P}(A,B)$ is closed under taking A-syzygies, then the finitistic dimension of B is finite.*

Note that the algebra B in the above corollary does not have to be monomial. So the corollary seems not to be obvious. The following simple example shows that a non-monomial algebra B can be embedded into a monomial algebra A such that $\mathrm{rad}(B)$ is an ideal in A and that $\mathscr{P}(A,B)$ is closed under taking A-syzygies.

Let A be the path algebra given by the quiver

$$\overset{1}{\circ} \xrightarrow{\ \alpha\ } \overset{2}{\circ} \xrightarrow{\ \beta\ } \overset{3}{\circ} \xrightarrow{\ \gamma\ } \overset{4}{\circ}.$$

We take B to be the subalgebra of A spanned by the primitive idempotent elements e_i with $1 \leq i \leq 4$ corresponding to the vertices of the quiver, together with the paths $\alpha, \gamma, \alpha\beta, \beta\gamma$ and $\alpha\beta\gamma$. Here, we write the composition of α with β as $\alpha\beta$. Then the algebra B is not monomial. However, one can check that $\mathscr{P}(A, B)$ is closed under taking A-syzygies. Note that $\mathscr{P}(A, B)$ is closed neither under extensions, nor under kernels of surjective homomorphisms, nor under cokernels of injective homomorphisms.

As another consequence of Corollary 1.2, we have the following result on the finitistic dimensions of Hochschild extensions of Artin R-algebras. For an A–A-bimodule ${}_AM_A$ and a Hochschild 2-cocycle $\alpha : A \otimes_R A \to M$, we denote by $H_\alpha(A, M)$ the Hochschild extension of A by M via α. For the precise definition of Hochschild extensions, one can find in [14] or [18].

Corollary 2.15. *Let B be a subalgebra of an Artin R-algebra A such that* $\mathrm{rad}(B) = \mathrm{rad}(A)$. *Suppose that ${}_AM_A$ is an A–A-bimodule and $\alpha : A \otimes_R A \to M$ is a Hochschild 2-cocycle. If* $\mathrm{fin.dim}(H_\alpha(A, M)) < \infty$, *then* $\mathrm{fin.dim}(H_\alpha(B, M)) < \infty$.

Proof. Given a Hochschild 2-cocycle $\alpha : A \otimes_R A \to M$, we may get an induced Hochschild 2-cocycle $\alpha' : B \otimes_R B \to M$ by composition of the canonical map $B \otimes_R B \to A \otimes_R A$ with α, which is denoted by α again by abuse of notation. Thus the Hochschild extension of B by the B–B-bimodule M via α is defined. The radical of $H_\alpha(B, M)$ is $\mathrm{rad}(B) \oplus M$, which is also the radical of $H_\alpha(A, M)$. Thus Corollary 2.15 follows from Corollary 1.2 immediately. $\square$

The next corollary deals with the finitistic dimensions of algebras of the form eBe with e an idempotent element in B. Recall that the *representation dimension* of A is defined to be the minimum of the global dimensions of algebras of the form $\mathrm{End}({}_AA \oplus D(A) \oplus M)$ with $M \in A\text{-mod}$. For further information on representation dimensions, we refer to [1] (see also [25] as well as the references therein).

Corollary 2.16. *Let B be a subalgebra of an Artin algebra A such that* $\mathrm{rad}(B) = \mathrm{rad}(A)$. *Suppose that e is an idempotent element in B such that the representation dimension of A/AeA is at most* 3. *If* $\mathrm{gl.dim}(A) \leq 4$, *then* $\mathrm{fin.dim}(eBe) < \infty$.

Proof. Under the above assumptions, we have $\mathrm{fin.dim}(eAe) < \infty$ by [26, Theorem 1.1]. Since $\mathrm{rad}(eBe) = e\mathrm{rad}(B)e = e\mathrm{rad}(A)e = \mathrm{rad}(eAe)$, the corollary follows from Theorem 1.1 applied to the extension $eBe \subseteq eAe$. $\square$

Now, let us make a few remarks on Theorem 1.1.

Remark. (1) Theorem 1.1(1) can be reformulated more generally as follows: Let $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_m$ be a finite chain of Artin algebras such that $\mathrm{rad}(A_i)$ is a left ideal in A_{i+1} and $\mathscr{P}(A_{i+1}, A_i)$ is closed under taking A_{i+1}-syzygies for each i. If $\mathrm{fin.dim}(A_m)$ is finite, then $\mathrm{fin.dim}(A_0)$ is finite.

(2) Suppose that $B \subseteq A$ is a radical-equal extension of Artin algebras. Though Theorem 1.1(2) gives an inequality for global dimensions of A and B, we do not have a similar inequality for finitistic dimensions. In fact, the difference of the finitistic dimensions of A and B could be any positive integer. For example, take an algebra A of global dimension $n \geq 1$, which is given by a quiver with relations, and glue all vertices of the quiver of A together to obtain a subalgebra B of A with the same radical. In this case, $\mathrm{gl.dim}(B) = \infty$, $\mathrm{fin.dim}(B) = 0$ and $\mathrm{fin.dim}(A) = n = \mathrm{gl.dim}(A)$.

(3) If an extension $B \subseteq A$ with $\mathrm{rad}(B)$ a left ideal in A is semisimple, then $\mathrm{gl.dim}(A) \leq \mathrm{gl.dim}(B)$ by Theorem 1.1. However, this inequality cannot be improved to equality "$\mathrm{gl.dim}(A) = \mathrm{gl.dim}(B)$". For example, if A is the path algebra of the quiver $\circ \to \circ \to \circ$, and if we glue the source vertex with the sink one in the quiver, then we get a subalgebra B of A with the same radical. Clearly, $\mathrm{gl.dim}(A) = 1 \neq 2 = \mathrm{gl.dim}(B)$. On the other hand, the upper bound "$\mathrm{gl.dim}(A) \leq \mathrm{gl.dim}(B)$" is optimal. For instance, if we take A to be the algebra given by the quiver $\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ \to \circ$ with relation $\alpha\beta = 0$, and B to be the gluing of the sink vertex with the ending vertex of α, then both A and B have global dimension 2.

(4) In [9] it was shown that under the assumptions of Corollary 1.2 the representation dimension of B is at most 3 if the representation dimension of A is at most 2. Thus $\mathrm{fin.dim}(B)$ is finite, according to a result of Igusa–Todorov [16]. Hence Corollary 1.2 can also be seen as a generalization of the result in [9].

(5) Recall that a full subcategory of A-mod is called *resolving* if it contains all projective modules in A-mod, and is closed under extensions and kernels of surjective homomorphisms. If A_B is projective for an extension $B \subseteq A$ of Artin R-algebras, then $\mathscr{P}(A, B)$ is a resolving subcategory in A-mod if and only if $\mathscr{P}(A, B)$ is closed under extensions (see [17, Proposition 7.6]). In particular, $\mathscr{P}(A, B)$ is closed under kernels of surjective homomorphisms if $\mathscr{P}(A, B)$ is closed under extensions. Another example for $\mathscr{P}(A, B)$ to be closed under kernels of surjective homomorphisms between (A, B)-projective modules is that there are injective B-modules ${}_BI_i$ and projective right B-modules P_i such that $A/B \simeq \bigoplus_i I_i \otimes_R P_i$ as B-bimodules. In this case, it was shown in [17, Proposition 7.5(a)] that $\mathscr{P}(A, B)$ is resolving.

In general, we have the following result.

Lemma 2.17. *Suppose that $\mathscr{P}(A, B)$ is closed under extensions. Then $\mathscr{P}(A, B)$ is closed under taking A-syzygies if and only if $\mathscr{P}(A, B)$ is closed under kernels of surjective homomorphisms, namely $\mathscr{P}(A, B)$ is resolving.*

Proof. Suppose that $\mathscr{P}(A,B)$ is closed under taking A-syzygies. Let $f : Y \to Z$ be a surjective homomorphism in $\mathscr{P}(A,B)$. The following pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega(Z) \arrow[r, equal] \arrow[d] & \Omega(Z) \arrow[d] & \\
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & P(Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

shows that $\mathrm{Ker}(f)$ is a direct summand of U which is in $\mathscr{P}(A,B)$ by assumption. Here $P(Z)$ denotes a projective cover of ${}_AZ$. Since $\mathscr{P}(A,B)$ is closed under direct summands, $\mathrm{Ker}(f) \in \mathscr{P}(A,B)$. Thus $\mathscr{P}(A,B)$ is closed under kernels of surjective homomorphisms. □

From the above lemma and Theorem 1.1 we have the following corollary.

Corollary 2.18. *Let $B \subseteq A$ be an extension of Artin algebras such that* $\mathrm{rad}(B)$ *is a left ideal in A and that $\mathscr{P}(A,B)$ is closed under extensions. If $\mathscr{P}(A,B)$ is resolving and if* $\mathrm{fin.dim}(A)$ *is finite, then* $\mathrm{fin.dim}(B)$ *is finite.*

Note that if A_B is projective for an extension $B \subseteq A$ of Artin algebras, then $\mathscr{P}(A,B)$ is closed under taking A-syzygies. In fact, for an A-module X in $\mathscr{P}(A,B)$, we see that X is a direct summand of $A \otimes_B X$, and therefore $\Omega_A(X)$ is a direct summand of $\Omega_A(A \otimes_B X)$. Since A_B is projective, $\Omega_A(A \otimes_B X)$ is a direct summand of $A \otimes_B \Omega_B(X)$ which lies in $\mathscr{P}(A,B)$ by Lemma 2.4(1). Note that $\mathscr{P}(A,B)$ is closed under direct summands. Thus $\Omega_A(X) \in \mathscr{P}(A,B)$.

Now, we point out a non-trivial example to show the existence of an n-hereditary extension which is not $(n-1)$-hereditary.

Let A be the algebra defined by the quiver

$$0\circ \xleftarrow{\alpha_0} \circ \xleftarrow{\alpha_1} \circ \xleftarrow{\alpha_2} \circ \cdots \circ \xleftarrow{\alpha_n} \circ n+1$$

with relations: $\alpha_n\alpha_{n-1} = \cdots = \alpha_2\alpha_1 = 0$. Let B be the subalgebra of A generated by α_0 and the primitive idempotent elements of A corresponding to the vertices of the quiver. The Auslander–Reiten quiver of this algebra can be drawn as follows:

$$\begin{tikzcd}[column sep=small]
 & & {\begin{smallmatrix}2\\1\\0\end{smallmatrix}} \arrow[dr] & & & & & & & & & & \\
 & {\begin{smallmatrix}1\\0\end{smallmatrix}} \arrow[ur] \arrow[dr] & & {\begin{smallmatrix}2\\1\end{smallmatrix}} \arrow[dr] & & & & \cdots \arrow[dr] & & & & {\begin{smallmatrix}n+1\\n\end{smallmatrix}} \arrow[dr] & \\
0 \arrow[ur] & & 1 \arrow[ur] & & 2 \arrow[dr] & & 3 \arrow[ur] & \cdots & n-1 \arrow[dr] & & n \arrow[ur] & & n+1 \\
 & & & & & {\begin{smallmatrix}3\\2\end{smallmatrix}} \arrow[ur] & & \cdots & & {\begin{smallmatrix}n\\n-1\end{smallmatrix}} \arrow[ur] & & &
\end{tikzcd}$$

This algebra has $2n+4$ non-isomorphic indecomposable modules. If $2 \leq i \leq n+1$, then the simple A-module S_i corresponding to the vertex i is not (A,B)-projective. Using the formula $\mathrm{Hom}_A(A \otimes_B X, I) \simeq \mathrm{Hom}_B(X, I)$ with I an indecomposable injective A-module, we can check the composition factors of the module $A \otimes_B X$. Thus the indecomposable A-module ${\begin{smallmatrix}2\\1\end{smallmatrix}}$ with the composition factors $\{S_2, S_1\}$ is not (A,B)-projective. All other indecomposable modules are (A,B)-projective. Hence there are $n+3$ indecomposable (A,B)-projective A-modules. The exact sequence

$$0 \to 2 \to \begin{matrix}3\\2\end{matrix} \to \cdots \to \begin{matrix}n\\n-1\end{matrix} \to \begin{matrix}n+1\\n\end{matrix}$$

shows that the extension $B \subseteq A$ is not $(n-1)$-hereditary. Clearly, there is only one exact sequence of length n:

$$0 \to \begin{matrix}1\\0\end{matrix} \to {\begin{smallmatrix}2\\1\\0\end{smallmatrix}} \to \begin{matrix}3\\2\end{matrix} \to \cdots \to \begin{matrix}n\\n-1\end{matrix} \to \begin{matrix}n+1\\n\end{matrix}$$

with the last n terms being (A,B)-projective. Hence the extension $B \subseteq A$ is n-hereditary, but not $(n-1)$-hereditary.

At the end of this section, let us reformulate the finitistic dimension conjecture for algebras over a perfect field in terms of relative global dimensions.

Let $B \subseteq A$ be an extension of Artin algebras with $\mathrm{rad}(B)$ a left ideal in A and $\mathrm{fin.dim}(A) < \infty$. Then Theorem 1.1 shows that if $\mathrm{gl.dim}(A,B) = 0$, then $\mathrm{fin.dim}(B) < \infty$.

Now we consider the following statement:

$(\star)$ Let $B \subseteq A$ be an extension of Artin algebras with $\mathrm{rad}(B)$ a left ideal in A and $\mathrm{fin.dim}(A) < \infty$. If $\mathrm{gl.dim}(A,B) \leq 1$, then $\mathrm{fin.dim}(B) < \infty$.

For finite-dimensional algebras over a perfect field, this statement is equivalent to the finitistic dimension conjecture, because we have the following observations.

Proposition 2.19. *Let $B \subseteq A$ be an extension of Artin algebras with* $\mathrm{rad}(B)$ *a left ideal in A. If this extension $B \subseteq A$ is radical-full, that is,* $\mathrm{rad}(A) = \mathrm{rad}(B)A$, *then* $\mathrm{gl.dim}(A,B) \leq 1$.

Proof. Let X be an A-module and $P_A(X)$ be a projective cover of X. We denote by K_X the kernel of the multiplication map $\mu_X : A \otimes_B X \to X$. Then we have an exact sequence of A-modules: $0 \to K_X \to A \otimes_B X \to X \to 0$. Now, we show that K_X is (A,B)-projective. In fact, we have the following commutative diagram of A-modules:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K_{P_A(X)} \arrow[r] \arrow[d] & K_X \arrow[d] & \\
 & A\otimes_B \Omega_A(X) \arrow[r] \arrow[d] & A\otimes_B P_A(X) \arrow[r] \arrow[d] & A\otimes_B X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_A(X) \arrow[r] \arrow[d] & P_A(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By the snake lemma, the map $K_{P_A(X)} \to K_X$ in the above diagram is a surjective A-homomorphism. By Lemma 2.1(3) (or the argument in the proof of Lemma 2.5), one can show that $(A/B)\otimes_B M \simeq (A/B)\otimes_B (M/\mathrm{rad}(_BM))$ for any B-module M. As $\mathrm{rad}(A) = \mathrm{rad}(B)A$, we know that $\mathrm{rad}(_AX) = \mathrm{rad}(_BX)$ for any A-module X. Since $K_X \simeq (A/B)\otimes_B X$ as B-modules (see Lemma 2.3), we have

$$\begin{aligned}
{}_BK_{P_A(X)} &\simeq (A/B)\otimes_B P_A(X) \simeq (A/B)\otimes_B \big(P_A(X)/\mathrm{rad}(B)P_A(X)\big)\\
&\simeq (A/B)\otimes_B (X/\mathrm{rad}(B)X) \simeq (A/B)\otimes_B X \simeq {}_BK_X.
\end{aligned}$$

Thus the surjective homomorphism $K_{P_A(X)} \to K_X$ is an isomorphism. Since $K_{P_A(X)}$ is (A,B)-projective, we see that K_X is (A,B)-projective. It is easy to see from the adjunction isomorphism $\mathrm{Hom}_A(A\otimes_B Y, X) \simeq \mathrm{Hom}_B(Y,X)$ that, for any A-module of the form $A\otimes_B Y$ with $Y \in B$-mod, the sequence

$$0 \to \mathrm{Hom}_A(A\otimes_B Y, K_X) \to \mathrm{Hom}_A(A\otimes_B Y, A\otimes_B X) \to \mathrm{Hom}_A(A\otimes_B Y, X) \to 0$$

is exact. Thus, for any direct summand X' of $A\otimes_B Y$, the sequence

$$0 \to \mathrm{Hom}_A(X', K_X) \to \mathrm{Hom}_A(X', A\otimes_B X) \to \mathrm{Hom}_A(X', X) \to 0$$

is exact. Hence the relative projective dimension of X is at most 1, and $\mathrm{gl.dim}(A,B) \le 1$. $\square$

The next observation is that every finite-dimensional $\mathbb{K}$-algebra B over a perfect field $\mathbb{K}$ can be embedded into a finite-dimensional $\mathbb{K}$-algebra A such that $\mathrm{rad}(B)$ is a left ideal in A and the extension $B \subseteq A$ is radical-full.

Indeed, since $\mathbb{K}$ is perfect, we can write $B = S \oplus \mathrm{rad}(B)$ where S is a maximal semisimple subalgebra of B. Let $\overline{B} := B/\mathrm{rad}^{n-1}(B)$, where n is the nilpotency index of $\mathrm{rad}(B)$. We define a map

$$B \longrightarrow A := \begin{pmatrix} S & 0 \\ \mathrm{rad}(B) & \overline{B} \end{pmatrix}$$

by sending $b = s + x$ to $\left(\begin{smallmatrix} s & 0 \\ x & \bar{b} \end{smallmatrix}\right)$ for $s \in S, x \in \mathrm{rad}(B)$, where $\bar{b}$ stands for the image of $b \in B$ under the canonical surjection from B to $\overline{B}$. One can check that A is a $\mathbb{K}$-algebra and this map is an injective homomorphism of algebras. We identify B with its image in A. Then $\mathrm{rad}(B)$ is a left ideal in A and $\mathrm{rad}(A) = \mathrm{rad}(B)A$. Thus the extension $B \subseteq A$ is radical-full.

The final observation is the following result in [10, Corollary 4.21]: For any Artin algebras A and B, and for any B-A-bimodule ${}_BM_A$, we can form the triangular matrix algebra $\Lambda := \left(\begin{smallmatrix} A & 0 \\ M & B \end{smallmatrix}\right)$, and get

$$\mathrm{fin.dim}(\Lambda) \leq \mathrm{fin.dim}(A) + \mathrm{fin.dim}(B) + 1.$$

Thus, for finite-dimensional algebras over a perfect field, the finitistic dimension conjecture is equivalent to the above statement ($\star$).

3. Algebras with a Decomposition

In this section, we compare the finitistic dimension of an algebra with that of its subalgebras, and prove Theorem 1.5. The following definition is motivated by the dual extensions in [21, 22].

Definition 3.1. Let A be an Artin R-algebra, B, C and S be three subalgebras of A (with the same identity). We say that A decomposes into a twisted tensor product of B and C over S, denoted by $A = B \wedge C$, if

(1) S is a maximal semisimple subalgebra of A (that is, S is a semisimple R-algebra such that $A = S \oplus \mathrm{rad}(A)$ as a direct sum of R-modules) such that $B \cap C = S$.
(2) The multiplication map $\varphi : C \otimes_S B \simeq {}_CA_B$ is an isomorphism of C–B-bimodules.
(3) $\mathrm{rad}(B)\mathrm{rad}(C) \subseteq \mathrm{rad}(C)\mathrm{rad}(B)$.

From this definition, we see that if A decomposes into a twisted tensor product of B and C over S, then $B = S \oplus \mathrm{rad}(B)$ and $C = S \oplus \mathrm{rad}(C)$, and the three algebras A, B and C have a common complete set $\{e_1, \ldots, e_t\}$ of primitive orthogonal idempotent elements. But, in general, neither $\mathrm{rad}(B)$ nor $\mathrm{rad}(C)$ is a left ideal in A.

In the following, we develop some basic properties of twisted tensor products.

Lemma 3.2. *If $A = B \wedge C$, then*

(a) *${}_CA$ and A_B are projective.*
(b) *$E(i) \otimes_C A_B \simeq e_iB_B$, where $E(i)$ is the right simple C-module $e_iC/\mathrm{rad}(e_iC)$ with e_i a primitive idempotent element in C.*

Proof. (a) Since S is semisimple, B is projective as an S-module. Then ${}_CA \simeq {}_CC \otimes_S B$ is projective as C-modules. In the same way, we can show that A_B is projective.

(b) $E(i) \otimes_C A_B \simeq E(i) \otimes_C C \otimes_S B_B \simeq E(i) \otimes_S B_B \simeq e_i B_B$. $\square$

Lemma 3.3. *If $A = B \wedge C$, then* $\mathrm{rad}(C)B$ *is an ideal in A, and $A/\mathrm{rad}(C)B \simeq B$ as algebras. Moreover, every B-module can be regarded as an A-module via this isomorphism, and the isomorphism in Lemma* 3.2(b) *then is an isomorphism of right A-modules.*

The following lemma is a special case of a result in [19, Lemma 1].

Lemma 3.4. *Let A be an Artin algebra, I be a nilpotent ideal of A, and X be an A-module. If $\mathrm{Tor}_p^A(A/I, X) = 0$ for all $p \geq 1$, then* $\mathrm{proj.dim}({}_AX) = \mathrm{proj.dim}({}_{A/I}(X/IX))$.

Lemma 3.5. *Let A be an Artin algebra and I be a nilpotent ideal of A. Suppose* $\mathrm{fin.dim}(A/I) = m$. *If there is a non-negative integer n such that $\mathrm{Tor}_k^A(A/I, X) = 0$ for all $k > n$ and all A-modules X with $\mathrm{proj.dim}({}_AX) < \infty$, then $\mathrm{fin.dim}(A) \leq m + n$.*

Proof. Take an A-module X with $\mathrm{proj.dim}({}_AX) < \infty$. We may assume that $\mathrm{proj.dim}({}_AX) = s > n$. Since $\mathrm{Tor}_k^A(A/I, X) = 0$ for all $k > n$, we have $\mathrm{Tor}_k^A(A/I, \Omega_A^n(X)) = 0$ for all $k > 0$. By Lemma 3.4, $\mathrm{proj.dim}({}_A\Omega_A^n(X)) = \mathrm{proj.dim}({}_{A/I}(\Omega_A^n(X)/I\Omega_A^n(X)))$. Thus $\mathrm{proj.dim}({}_{A/I}(\Omega_A^n(X)/I\Omega_A^n(X)))$ is finite, and therefore $\mathrm{proj.dim}({}_{A/I}(\Omega_A^n(X)/I\Omega_A^n(X))) \leq m$. This means that $s - n = \mathrm{proj.dim}({}_A\Omega_A^n(X)) \leq m$. It follows that $\mathrm{fin.dim}(A) \leq m + n$. $\square$

Lemma 3.6. *Suppose $A = B \wedge C$. If $\mathrm{fin.dim}(C) = n < \infty$, then $\mathrm{Tor}_p^A(B, X) = 0$ for all $p > n$ and all A-modules X with $\mathrm{proj.dim}({}_AX) < \infty$.*

Proof. Let X be an A-module with $\mathrm{proj.dim}({}_AX) < \infty$. We may assume $\mathrm{proj.dim}({}_AX) = s > n$. Take a minimal projective resolution of ${}_AX$:

$$0 \longrightarrow P_s \xrightarrow{f_s} P_{s-1} \xrightarrow{f_{s-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0.$$

By restricting to C, the above sequence provides a projective resolution of the C-module ${}_CX$ by Lemma 3.2(a). So we have $\mathrm{proj.dim}({}_CX) \leq s < \infty$. Since $\mathrm{fin.dim}(C) = n$, we have $\mathrm{proj.dim}({}_CX) = k \leq n$. Thus ${}_C\Omega_A^k(X)$ is projective, and we have a split exact sequence of C-modules:

$$0 \longrightarrow P_s \xrightarrow{f_s} P_{s-1} \xrightarrow{f_{s-1}} \cdots \xrightarrow{f_{k+1}} P_k \xrightarrow{f_k} \Omega_A^k(X) \longrightarrow 0.$$

Therefore the following sequence

$$0 \to E(i) \otimes_C P_s \to E(i) \otimes_C P_{s-1} \to \cdots \to E(i) \otimes_C P_k \to E(i) \otimes_C \Omega_A^k(X) \to 0$$

is exact, where $E(i)$ is the right simple C-module $e_iC/\mathrm{rad}(e_iC)$ with e_i a primitive idempotent element in C. As $e_iB \otimes_A P_j \simeq E(i) \otimes_C A \otimes_A P_j \simeq E(i) \otimes_C P_j$ for all

$0 \le j \le s$, $1 \le i \le t$ by Lemmas 3.2 and 3.3, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(i)\otimes_C P_s & \longrightarrow \cdots \longrightarrow & E(i)\otimes_C P_k & \longrightarrow & E(i)\otimes_C \Omega_A^k(X) & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
0 & \longrightarrow & E(i)\otimes_C A\otimes_A P_s & \longrightarrow \cdots \longrightarrow & E(i)\otimes_C A\otimes_A P_k & \longrightarrow & E(i)\otimes_C A\otimes_A \Omega_A^k(X) & \longrightarrow & 0 \\
 & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
0 & \longrightarrow & e_iB\otimes_A P_s & \longrightarrow \cdots \longrightarrow & e_iB\otimes_A P_k & \longrightarrow & e_iB\otimes_A \Omega_A^k(X) & \longrightarrow & 0
\end{array}$$

Thus the exactness of the bottom row implies that $\mathrm{Tor}_p^A(e_iB, X) = 0$ for all $p > k$. Consequently, $\mathrm{Tor}_p^A(B, X) = 0$ for all $p > n \ge k$. □

Proof of Theorem 1.5. By Lemma 3.3, we have an algebra isomorphism $A/\langle \mathrm{rad}(C)\rangle \simeq B$, where $\langle \mathrm{rad}(C)\rangle$ denotes the ideal of A generated by $\mathrm{rad}(C)$. Thus $\mathrm{fin.dim}(A/\langle \mathrm{rad}(C)\rangle) = \mathrm{fin.dim}(B) = m$. Note that the algebra isomorphism is also an isomorphism of right A-modules.

Since $\mathrm{rad}(B)\mathrm{rad}(C) \subseteq \mathrm{rad}(C)\mathrm{rad}(B)$, we have

$$\langle \mathrm{rad}(C)\rangle = A\mathrm{rad}(C)A = \mathrm{rad}(C) + \mathrm{rad}(C)\mathrm{rad}(B) = \mathrm{rad}(C)B.$$

This is a nilpotent ideal of A. If X is an A-module with $\mathrm{proj.dim}({}_AX) < \infty$, then $\mathrm{Tor}_p^A(B, X) = 0$ for all $p > n$ by Lemma 3.6, that is, $\mathrm{Tor}_p^A(A/\langle \mathrm{rad}(C)\rangle, X) = 0$ for all $p > n$. Consequently, $\mathrm{fin.dim}(A) \le m + n$ by Lemma 3.5.

To see that $m \le \mathrm{fin.dim}(A)$, we shall show that if $f : P \to X$ is a projective cover of the B-module X, then $1\otimes_B f : A\otimes_B P \to A\otimes_B X$ is a projective cover of the A-module $A \otimes_B X$. Once this is proved, it follows easily that $m \le \mathrm{fin.dim}(A)$ since $\mathrm{proj.dim}({}_AA \otimes_B X) = \mathrm{proj.dim}({}_BX)$ for all ${}_BX$ by the exactness of the functor $A \otimes_B -$.

In fact, since $A/\mathrm{rad}(C)B \simeq B$, we may view each B-module as an A-module via this isomorphism. Thus the multiplication map $\mu : A \otimes_B X \to X$ is a surjective homomorphism of A-modules. Since simple B-modules are also simple A-modules, the composition $(1 \otimes f)\mu : A \otimes_B P \to X$ is a projective cover of the A-module X. Thus we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(1\otimes f) & \longrightarrow & A\otimes_B P & \xrightarrow{1\otimes f} & A\otimes_B X & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \mu\downarrow & & \\
0 & \longrightarrow & \Omega_A(X) & \longrightarrow & A\otimes_B P & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

The first commutative square shows that $\mathrm{Ker}(1 \otimes f)$ can be embedded in $\Omega_A(X) \subseteq \mathrm{rad}(A\otimes_B P)$. Thus $A\otimes_B P \to A\otimes_B X$ is a projective cover of $A\otimes_B X$. This finishes the proof. □

As a consequence, we apply our result Theorem 1.5 to trivially twisted extensions [22].

Let $\mathbb{K}$ be a field. Let B be a $\mathbb{K}$-algebra given by a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with relations $\{\sigma_i \mid i \in I_0\}$, and let C be a $\mathbb{K}$-algebra given by another quiver $\Delta = (\Delta_0, \Delta_1)$

with relations $\{\tau_j \mid j \in J_0\}$. Assume that S_0 is a subset contained in $\Gamma_0 \cap \Delta_0$. Now we define a new algebra A, called the *trivially twisted extension* of B and C at S_0, in the following manner: A is given by the quiver $Q = (Q_0 := \Gamma_0 \,\dot\cup\, (\Delta_0 \backslash S_0), Q_1 := \Gamma_1 \,\dot\cup\, \Delta_1)$, with the relations $\{\sigma_i \mid i \in I_0\} \cup \{\tau_j \mid j \in J_0\} \cup \{\alpha\beta \mid \alpha \in \Gamma_1, \beta \in \Delta_1\}$, where $\alpha\beta$ means that α comes first and then β follows. Note that if $S_0 = \Delta_0 = \Gamma_0$, then we call A the *trivially twisted extension* of B and C; and if, in addition, C is the opposite algebra B^{op} of B, then A is called the *dual extension* of B. In this case we have $A \simeq A^{\mathrm{op}}$. However, the dual extension of B is usually not isomorphic to the dual extension of B^{op}.

Note that the trivially twisted extension A of B and C at S_0 modulo the ideal generated by $\{\beta\alpha \mid \alpha \in \Gamma_1, \beta \in \Delta_1\}$ in A is the pullback algebra D of B and C over $\mathbb{K}S_0$, along the canonical surjections. Thus we have the following result.

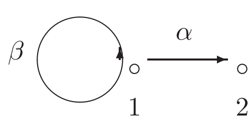
Corollary 3.7. *Let A be the trivially twisted extension of two finite-dimensional algebras B and C at S_0, and let D be the above-defined quotient algebra of A. If* $\mathrm{fin.dim}(B) < \infty$ *and* $\mathrm{fin.dim}(C) < \infty$, *then*

(1) $\mathrm{fin.dim}(A) \leq \mathrm{fin.dim}(B) + \mathrm{fin.dim}(C) < \infty$.
(2) $\mathrm{fin.dim}(D) < \infty$.

Proof. By definition, A decomposes into a twisted tensor product of B' and C' over a maximal semisimple subalgebra S', where $B' = B \oplus \mathbb{K}(\Gamma_0 \backslash \Delta_0)$, $C' = C \oplus \mathbb{K}(\Delta_0 \backslash \Gamma_0)$ and S' is the semisimple algebra over $\mathbb{K}$ generated by the union of Δ_0 and Γ_0. In this case, we have $\mathrm{rad}(B')\mathrm{rad}(C') = 0$. Thus, by Theorem 1.5, the statement (1) follows. The statement (2) follows from the fact that D can be embedded in $B \oplus C$ such that $\mathrm{rad}(D) = \mathrm{rad}(B) \oplus \mathrm{rad}(C)$. Thus $\mathrm{fin.dim}(D) < \infty$ by Corollary 1.2. $\square$

Remark. (1) In general, we cannot get "$\mathrm{fin.dim}(A) = \mathrm{fin.dim}(B) + \mathrm{fin.dim}(C)$" in Theorem 1.5. For example, let B and C be the algebra (over a field) of the quiver $\circ \to \circ$. Then the trivially twisted extension of B and C is the Kronecker algebra defined by the quiver $\circ \rightrightarrows \circ$. Clearly, $\mathrm{fin.dim}(A) = \mathrm{fin.dim}(B) = \mathrm{fin.dim}(C) = 1$. Thus $\mathrm{fin.dim}(A) \neq \mathrm{fin.dim}(B) + \mathrm{fin.dim}(C)$.

(2) For an Artin algebra A, we have $\mathrm{gl.dim}(A) = \mathrm{gl.dim}(A^{\mathrm{op}})$. Moreover, if A is the dual extension of an algebra B, then $\mathrm{gl.dim}(A) = 2 \cdot \mathrm{gl.dim}(B)$ (see [21] for details). All these, however, are no longer true for finitistic dimension. The following example shows that even if A is the dual extension of B, we cannot get $\mathrm{fin.dim}(A) = \mathrm{fin.dim}(B) + \mathrm{fin.dim}(B^{\mathrm{op}})$. Indeed, let B be the algebra defined by the following quiver with relations:

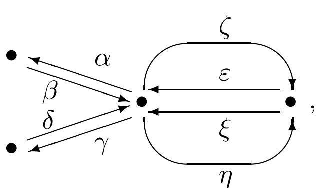


$\beta\alpha = \beta^2 = 0.$

Then $\text{fin.dim}(B) = 1$ and $\text{fin.dim}(B^{\text{op}}) = 0$. Let A be the dual extension of B. Since all indecomposable projective A-modules have the same Loewy length, the algebra A has finitistic dimension zero. Thus $\text{fin.dim}(A) \neq \text{fin.dim}(B) + \text{fin.dim}(B^{\text{op}})$. If A' is the dual extension of B^{op}, then $\text{fin.dim}(A') = 1$. This example shows also that the dual extension of B is not isomorphic to the dual extension of B^{op}.

Finally, we give an example to illustrate how our results can be used to estimate finitistic dimensions of algebras.

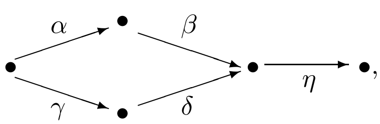
Let A be the following algebra (over a field) given by quiver with relations:



$$\alpha\beta = \gamma\delta,\ \beta\alpha = \beta\gamma = 0,\quad \varepsilon\eta = \xi\eta = \varepsilon\alpha = \varepsilon\gamma = 0,\quad \xi\alpha = \xi\gamma = \delta\alpha = \delta\gamma = 0,\quad \varepsilon\zeta = \zeta\varepsilon = \zeta\xi = 0.$$

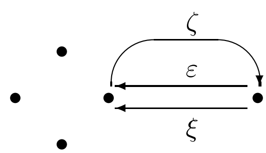
Now, we consider the trivially twisted extension A' of B and C, where B and C are as follows:

C



$$\alpha\beta = \gamma\delta.$$

B



$$\varepsilon\zeta = \zeta\varepsilon = \zeta\xi = 0.$$

Then we know from Corollary 3.7 that $\text{fin.dim}(A')$ is finite since both $\text{fin.dim}(B)$ and $\text{fin.dim}(C)$ are finite. Since A is obtained from A' by gluing vertices, we have $\text{rad}(A) = \text{rad}(A')$, and therefore $A \subseteq A'$ is a semisimple extension. Now, by Theorem 1.1, A has finite finitistic dimension.

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