New invariants of stable equivalences of algebras

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Abstract

We show that the Auslander-Reiten conjecture on stable equivalences holds true for principal centralizer matrix algebras over an arbitrary field and for Frobenius-finite algebras over an algebraically closed field, that stable equivalences of algebras with positive ν-dominant dimensions preserve stable equivalences of their Frobenius parts, and that the delooping levels, ϕ-dimensions and ψ-dimensions are invariants of stable equivalences of Artin algebras without nodes.

Contents

1 Introduction

Stable equivalence of algebras is one of the prominent equivalences in the representation theory of algebras and groups, and has been studied by many authors. For instance, Auslander and Reiten showed that any algebra with radical-square-zero is stably equivalent to a hereditary algebra [4]. Martínez-Villa showed that the global and dominant dimensions are invariants of stable equivalences of algebras with no nodes and no semisimple summands [26]. Considerable efforts notwithstanding, stable equivalences seem still to be understood. For example, a long-standing, unsolved problem is the famous Auslander-Reiten conjecture on stable equivalences, which says that stably equivalent Artin algebras have the same number of non-isomorphic, non-projective simple modules (see, for instance, [6, Conjecture (5), p.409], or [30, Conjecture 2.5]). So the conjecture predicts that the number of non-isomorphic, non-projective simple modules should be invariant under stable equivalences. Auslander and Reiten proved that if an Artin algebra is stably equivalent to a hereditary algebra then the two algebras have the same number of non-isomorphic, non-projective simple modules [4]. For algebras over algebraically closed field, the conjecture was verified for representation-finite algebras, and reduced to self-injective algebras without nodes (see [25, 26]). Recently, the conjecture is proved for stable equivalences of Morita type between Frobenius-finite algebras without semisimple summands (see [17]), and for stable equivalences between special biserial algebras (see [1], [28]). In general, the conjecture still remains open.

The purpose of this note is to prove the following results on stable equivalences of algebras. The first one is that the Auslander-Reiten conjecture holds true for principal centralizer matrix algebras over an arbitrary field and for Frobenius-finite algebras over an algebraically closed field. The second one is that stable equivalences of Artin algebras with positive ν-dominant dimensions preserve stable equivalences of their Frobenius parts, and the third one is that stable equivalences preserve the delooping levels, ϕ-dimensions and ψ-dimensions of Artin algebras without nodes.

In the following, let us describe our results more precisely.

Let A be an Artin algebra over a commutative Artin ring k. By A-mod we denote the category of all finitely generated left A-modules. Related to simple A-modules, Vincent Gélinas introduces recently the delooping levels of algebras in [14]. The significance of delooping levels is that the finitistic dimensions of algebras can be bounded by the delooping levels (see [14, Proposition 1.3]), thus providing a way to understand the unsolved finitistic dimension conjecture which states that Artin algebras always have finite finitistic dimension (see [7], or [6, Conjecture (11), p.410]). Recall that the *finitistic dimension* of A, denoted findim(A), is the supremum of the projective dimensions of modules $X \in A$-mod that have finite projective dimension. By definition, the *delooping level* of an A-module $X \in A$-mod, denoted del(X), is the smallest number $d \geq 0$ such that the d-th syzygy $\Omega^d(X)$ of X is a direct summand of a module of the form $P \oplus \Omega^{d+1}(M)$ for an A-module M and a projective A-module P, where Ω is the syzygy (or loop) operator of A. If such a number d does not exist, one defines del$(X) = \infty$. The *delooping level* of A, denoted del(A), is the maximum of the delooping levels of all non-isomorphic simple A-modules. Clearly, del$(X \oplus Y) = \max\{\text{del}(X), \text{del}(Y)\}$ for $X, Y \in A$-mod, and del$(A) = \text{del}(\text{top}(_AA))$, where top$(_AX)$ denotes the top of an A-module $_AX$. It was shown in [14] that findim$(A) \leq \text{del}(A^{\text{op}})$, where A^{op} stands for the opposite algebra of A.

To understand the finitistic dimensions of algebras, Igusa and Todorov introduced the ϕ- and ψ-dimensions for Artin algebras in [20]. Let $K(A)$ be the Grothendieck group of A, that is, the quotient of the free abelian group generated by the isomorphism classes $[X]$ with $X \in A$-mod, modulo the relations: (1) $[Z] = [X] + [Y]$ if $_AZ \simeq _AX \oplus _AY$; (2) $[P] = 0$ if $_AP$ is projective. Then $K(A)$ is the free abelian group generated by the isomorphism classes of non-projective indecomposable A-modules $X \in A$-mod. Now, we recall two functions ϕ and ψ from A-mod to $\mathbb{N}$, the set of natural numbers, defined in [20].

The syzygy functor $\Omega : A\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}$ on the stable module category $A\text{-}\underline{\text{mod}}$ induces a group homomorphism $\Omega : K(A) \to K(A)$ of abelian groups, given by $\Omega([X]) := [\Omega(X)]$. For $X \in A$-mod, let $\langle X \rangle$ be the $\mathbb{Z}$-submodule of $K(A)$ generated by the isomorphism classes of non-projective, indecomposable direct summands of X. Since the rank of the image $\Omega\langle X \rangle$ of $\langle X \rangle$ under Ω does not exceed the finite rank of $\langle X \rangle$, it follows from Fitting's Lemma that there exists a smallest nonnegative integer $\phi(X)$ such that $\Omega : \Omega^n\langle X \rangle \to \Omega^{n+1}\langle X \rangle$ is an isomorphism for all $n \geq \phi(X)$. Furthermore, let

$$\psi(X) := \phi(X) + \sup\{\text{projdim}(Y) \mid Y \text{ is a direct summand of } \Omega^{\phi(X)}(X), \text{ projdim}(Y) < \infty\}.$$

Then the ϕ-dimension and ψ-dimension of A are defined by

$$\phi\dim(A) := \sup\{\phi(X) \mid X \in A\text{-mod}\} \quad \text{and} \quad \psi\dim(A) := \sup\{\psi(X) \mid X \in A\text{-mod}\}.$$

According to [20, Lemma 0.3], $\psi(X) = \phi(X) = \text{projdim}(X)$ if projdim$(X) < \infty$. Thus

$$\text{findim}(A) \leq \phi\dim(A) \leq \psi\dim(A) \leq \text{gl.dim}(A),$$

where gl.dim(A) means the global dimension of A.

Following [24], a non-projective, non-injective simple A-module S is called a *node* of A if the middle term P of the almost split sequence starting at S, $0 \to S \to P \to \text{Tr}D(S) \to 0$, is projective, where D is the usual duality of Artin algebras and Tr stands for the transpose of modules.

Theorem 1.1. *If A and B are stably equivalent Artin algebras without nodes, then* del$(A) = $ del(B)*,* $\phi\dim(A) = \phi\dim(B)$ *and* $\psi\dim(A) = \psi\dim(B)$*.*

As a consequence of Theorems 1.1, we have the result (see Proposition 3.4).

Proposition 1.2. *If A is a finite-dimensional self-injective algebra over a field and X is an A-module, then*

$$\text{del}(\text{End}_A(A \oplus X)) = \text{del}(\text{End}_A(A \oplus D\text{Tr}(X)) \text{ and } \phi\dim(\text{End}_A(A \oplus X)) = \phi\dim(\text{End}_A(A \oplus D\text{Tr}(X)),$$

where End$_A(M)$ *stands for the endomorphism algebra of an A-module M.*

Recall that a projective A-module P is said to be ν-*stably projective* [17] if $\nu_A^i P$ is projective for all $i > 0$, where ν is the Nakayama functor of A. By A-stp we mean the full subcategory of A-mod consisting of all ν-stably projective A-modules. If X is an A-module such that $\text{add}(X) = A$-stp, then the endomorphism algebra of the A-module X is called a *Frobenius part* of A, which is self-injective (see [25], or [17, Lemma 2.7]) and unique up to Morita equivalence. Note that Frobenius parts of Artin algebras were first given by Martínez-Villa in different but equivalent terms in [25]. An Artin algebra is said to be *Frobenius-finite* if its Frobenius part is representation-finite. By [26, Theorem 2.6], over an algebraically closed field k, every stable equivalence of k-algebras with no nodes and no semisimple direct summands induces a stable equivalence of their Frobenius parts. This is also true for Artin algebras by checking the proof there. In this note we show that the assumption that algebras have no nodes and no semisimple direct summands can be replaced by positive ν-dominant dimensions (see Section 3.2 for definition).

Theorem 1.3. *Suppose that A and B are Artin algebras of ν-dominant dimension at least* 1*. If A and B are stably equivalent, then so are their Frobenius parts.*

For a positive integer n, let $M_n(A)$ be the $n \times n$ matrix algebra over A. Given a nonempty subset C of $M_n(A)$, the *centralizer* algebra of C in $M_n(A)$ is defined by

$$S_n(C,A) := \{a \in M_n(A) \mid ca = ac, \, \forall \, c \in C\}.$$

If $C = \{c\}$ consists of only a single matrix c, we write $S_n(c,A)$ for $S_n(\{c\},A)$ and call $S_n(c,A)$ a *principal centralizer matrix algebra*. Clearly, $S_n(C,A) = \cap_{c \in C} S_n(c,A)$. If C consists of invertible matrices, $S_n(C,A)$ is nothing else than the invariant algebra under the action of conjugation. Particularly, centralizer matrix algebras include algebras of centrosymmetric matrices which have applications in engineering problems and quantum physics [10], and the Auslander algebras of the truncated polynomial algebras (see [32]), which are quasi-hereditary and useful in the study of unipotent radicals of reductive groups. For further homological properties of principal centralizer matrix algebras, we refer to [33].

Theorem 1.4. *The Auslander-Reiten conjecture on stable equivalences holds true for the following finite-dimensional algebras:*
  (1) *Principal centralizer matrix algebras over a field.*
  (2) *Frobenius-finite algebras over an algebraically closed field.*

If we assume in Theorem 1.4(2) that the stable equivalences are of Morita type and both algebras have no semisimple direct summands, then Theorem 1.4(2) follows from [17, Theorem 1.1]. Since Frobenius-finite algebras properly contain representation-finite algebras, Theorem 1.4(2) generalizes also a result in [25, Theorem 3.4] which states that the Auslander-Reiten conjecture on stable equivalences holds true for representation-finite algebras over an algebraically closed field.

This paper is structured as follows. In Section 2, we recall basic facts on stable equivalences. In Section 3, we prove all results mentioned in Introduction. In the course of our proofs, we point out that almost ν-stable derived equivalences preserve the delooping levels, ϕ-dimensions and ψ-dimensions of algebras over a field (see Proposition 3.4), though derived equivalences in general may not have this property. We also conjecture that the finiteness of delooping levels of Artin algebras is invariant under derived equivalences.

2   Preliminaries

In this section, we fix notations and recall basic results of stable equivalences.

Let A be an Artin algebra over a commutative Artin ring k. By A-mod we denote the category of all finitely generated left A-modules. Let A^{op} be the opposite algebra of A, we understand a right A-module as a

left A^{op}-module. We denote by D the usual duality of Artin algebra from A-mod to A^{op}-mod. For $M \in A$-mod, let $\Omega_A(M)$ be the syzygy of $_A M$; $\mathrm{Tr}(M)$ the transpose of M, which is an A^{op}-module; and add(M) the full additive subcategory of A-mod consisting of all direct summands of finite sums of copies of M. We write $\mathrm{End}_A(M)$ for the endomorphism algebra of $_A M$.

Let A-mod$_{\mathscr{P}}$ (respectively, A-mod$_{\mathscr{I}}$) be the full subcategory of A-mod consisting of those modules that do not have nonzero projective (respectively, injective) direct summands. Let $\mathscr{P}(A)_{\mathscr{I}}$ (respectively, $\mathscr{I}(A)_{\mathscr{P}}$) denote the set of all isomorphism classes of indecomposable projective (respectively, injective) A-modules without any nonzero injective (respectively, projective) direct summands.

The stable category A-$\underline{\mathrm{mod}}$ of A has the same objects as A-mod, its morphism set $\underline{\mathrm{Hom}}_A(X,Y)$ of two modules X and Y is the quotient k-module of $\mathrm{Hom}_A(X,Y)$ modulo all homomorphisms that factorize through a projective A-module. For $f \in \mathrm{Hom}_A(X,Y)$, we write $\underline{f}$ for the image of f in $\underline{\mathrm{Hom}}_A(X,Y)$. Note that $X \simeq Y$ in A-$\underline{\mathrm{mod}}$ if and only if there are two projective modules $P, Q \in A$-mod such that $X \oplus P \simeq Y \oplus Q$ as A-modules. In this case, $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus Y)$ are Morita equivalent.

Let $(A$-$\underline{\mathrm{mod}})$-mod be the category of all finitely presented functors from $(A$-$\underline{\mathrm{mod}})^{\mathrm{op}}$ to the category $\mathscr{A}$ of all abelian groups. Recall that a functor $H : (A$-$\underline{\mathrm{mod}})^{\mathrm{op}} \to \mathscr{A}$ is said to be *finitely presented* if there exists an exact sequence of functors $\underline{\mathrm{Hom}}_A(-,X) \to \underline{\mathrm{Hom}}_A(-,Y) \to H \to 0$ with X and Y in A-$\underline{\mathrm{mod}}$. It was known from [2] that $(A$-$\underline{\mathrm{mod}})$-mod is an abelian category, its projective objects are precisely the functors $\underline{\mathrm{Hom}}_A(-,X)$ for $X \in A$-mod$_{\mathscr{P}}$, and its injective objects of $(A$-$\underline{\mathrm{mod}})$-mod are precisely the functors $\mathrm{Ext}_A^1(-,X)$ for $X \in A$-mod$_{\mathscr{P}}$.

Artin algebras A and B over a commutative Artin ring k are said to be *stably equivalent* if the two stable categories A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$ are equivalent as k-categories.

Assume that $F : A$-$\underline{\mathrm{mod}} \to B$-$\underline{\mathrm{mod}}$ is an equivalence of k-categories with a quasi-inverse functor $G : B$-$\underline{\mathrm{mod}} \to A$-$\underline{\mathrm{mod}}$. So F and G are additive functors and induce two equivalences α and β of abelian categories (see [3, Section 8])

$$\alpha : (A\text{-}\underline{\mathrm{mod}})\text{-mod} \xrightarrow{\simeq} (B\text{-}\underline{\mathrm{mod}})\text{-mod} \quad \text{and} \quad \beta : (B\text{-}\underline{\mathrm{mod}})\text{-mod} \xrightarrow{\simeq} (A\text{-}\underline{\mathrm{mod}})\text{-mod},$$

and two one-to-one correspondences

$$F : A\text{-mod}_{\mathscr{P}} \longleftrightarrow B\text{-mod}_{\mathscr{P}} : G \quad \text{and} \quad F' : A\text{-mod}_{\mathscr{I}} \longleftrightarrow B\text{-mod}_{\mathscr{I}} : G'$$

such that
$$\alpha(\underline{\mathrm{Hom}}_A(-,X)) \simeq \underline{\mathrm{Hom}}_B(-,F(X)) \quad \text{and} \quad \alpha(\mathrm{Ext}_A^1(-,Y)) \simeq \mathrm{Ext}_B^1(-,F'(Y)),$$
$$\beta(\underline{\mathrm{Hom}}_B(-,U)) \simeq \underline{\mathrm{Hom}}_A(-,G(U)) \quad \text{and} \quad \beta(\mathrm{Ext}_B^1(-,V)) \simeq \mathrm{Ext}_A^1(-,G'(V)),$$

for $X \in A$-mod$_{\mathscr{P}}$, $Y \in A$-mod$_{\mathscr{I}}$, $U \in B$-mod$_{\mathscr{P}}$ and $V \in B$-mod$_{\mathscr{I}}$. For convenience, we set $F(P) = 0$ for a projective module P, and $F'(I) = 0$ for an injective module I.

The following lemma is useful for our later discussions.

Lemma 2.1. ([3, Section 7, p.347]) *If* $X,Y \in A$-mod$_{\mathscr{I}}$, *then* $X \simeq Y$ *in* A-$\underline{\mathrm{mod}}$ *if and only if* $\mathrm{Ext}_A^1(-,X) \simeq \mathrm{Ext}_A^1(-,Y)$ *in* $(A$-$\underline{\mathrm{mod}})$-mod.

A node S of A is called an *F-exceptional node* if $F(S) \not\simeq F'(S)$. By $\mathfrak{n}_F(A)$ we denote the set of isomorphism classes of F-exceptional nodes of A. By [5, Lemma 3.4], if X is a non-injective, non-projective, indecomposable A-module, then $F(X) \simeq F'(X)$. Thus $\mathfrak{n}_F(A)$ coincides with the set of isomorphism classes of non-projective, non-injective, indecomposable A-modules X such that $F(X) \not\simeq F'(X)$.

In the following, let

$$\triangle_A := \mathfrak{n}_F(A) \dot{\cup} \mathscr{P}(A)_{\mathscr{I}} \quad \text{and} \quad \triangledown_A := \mathfrak{n}_F(A) \dot{\cup} \mathscr{I}(A)_{\mathscr{P}},$$

where $\dot{\cup}$ stands for the disjoint union of sets; $\mathscr{P}(A)_{\mathscr{I}}$ (respectively, $\mathscr{I}(A)_{\mathscr{P}}$) stands for the set of all i-somorphism classes of indecomposable projective (respectively, injective) A-modules without any nonzero

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injective (respectively, projective) summands. By $\triangle_A^c$ we mean the class of indecomposable, non-injective A-modules which do not belong to $\triangle_A$. Thus each module $M \in A\text{-mod}_{\mathscr{I}}$ admits a unique decomposition (up to isomorphism)

$$M \simeq M_1 \oplus M_2$$

with $M_1 \in \text{add}(\triangle_A)$ and $M_2 \in \text{add}(\triangle_A^c)$.

Lemma 2.2. ([9, Lemma 4.10(1)]) *The functor F induces the bijections*

$$F : \bigtriangledown_A \longleftrightarrow \bigtriangledown_B : G, \quad F' : \triangle_A \longleftrightarrow \triangle_B : G' \quad and \quad F' : \triangle_A^c \longleftrightarrow \triangle_B^c : G'.$$

An exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in A-mod is called *minimal* if it does not have a split exact sequence as its direct summand, that is, there do not exist isomorphisms u, v, w such that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad g \quad} & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\
0 & \longrightarrow & X_1 \oplus X_2 & \xrightarrow{\left(\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right)} & Y_1 \oplus Y_2 & \xrightarrow{\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right)} & Z_1 \oplus Z_2 & \longrightarrow & 0
\end{array}
$$

is exact commutative in A-mod, where $Y_2 \neq 0$ and the sequence $0 \to X_2 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z_2 \to 0$ splits. By definition, a minimal exact sequence does not split.

Lemma 2.3. ([3, Theorem 7.5] or [5, Proposition 2.1]) *Let $H \in (A\text{-}\underline{\text{mod}})\text{-mod}$ and $0 \to X \to Y \to Z \to 0$ be a minimal exact sequence in A-mod such that the induced sequence*

$$0 \longrightarrow \underline{\text{Hom}}_A(-,X) \longrightarrow \underline{\text{Hom}}_A(-,Y) \longrightarrow \underline{\text{Hom}}_A(-,Z) \longrightarrow H \longrightarrow 0$$

of functors is exact. Then the following hold.
    (1) *The induced exact sequence of functors*

$$\underline{\text{Hom}}_A(-,Y) \longrightarrow \underline{\text{Hom}}_A(-,Z) \longrightarrow H \longrightarrow 0$$

is a minimal projective presentation of H in $(A\text{-}\underline{\text{mod}})\text{-mod}$.
    (2) *The induced exact sequence of functors*

$$0 \longrightarrow H \longrightarrow \text{Ext}_A^1(-,X) \longrightarrow \text{Ext}_A^1(-,Y)$$

is a minimal injective copresentation of H in $(A\text{-}\underline{\text{mod}})\text{-mod}$.

The following lemma is from [26, Lemma 1.6], while its proof is referred to [5].

Lemma 2.4. *If $H \in (A\text{-}\underline{\text{mod}})\text{-mod}$ has a minimal projective presentation*

$$\underline{\text{Hom}}_A(-,Y) \xrightarrow{\underline{\text{Hom}}_A(-,g)} \underline{\text{Hom}}_A(-,Z) \longrightarrow H \longrightarrow 0$$

with $Y,Z \in A\text{-mod}_{\mathscr{P}}$, then there is a minimal exact sequence

$$0 \longrightarrow X \longrightarrow Y \oplus P \xrightarrow{g'} Z \longrightarrow 0$$

in A-mod, where $\underline{g}' = \underline{g}$ in $A\text{-}\underline{\text{mod}}$ and P is a projective A-module.

The following generalization of [26, Theorem 1.7] shows that the functor F possesses certain "exactness" property.

Lemma 2.5. *Let $0 \to X \oplus X' \to Y \oplus \bar{Y} \oplus I \oplus P \oplus P' \xrightarrow{g} Z \to 0$ be a minimal exact sequence in A-mod with $X, Y \in \mathrm{add}(\triangle_A^c)$, $X' \in \mathrm{add}(\triangle_A)$, $\bar{Y} \in \mathrm{add}(\mathfrak{n}_F(A))$, $I \in \mathrm{add}(\mathscr{I}(A)_{\mathscr{P}})$, $P \in \mathrm{add}(\mathscr{P}(A)_{\mathscr{I}})$, $P' \in A$-prinj and $Z \in A\text{-mod}_{\mathscr{P}}$. Then there exists a minimal exact sequence*

$$0 \longrightarrow F(X) \oplus F'(X') \longrightarrow F(Y \oplus \bar{Y} \oplus I) \oplus Q \oplus Q' \xrightarrow{g'} F(Z) \longrightarrow 0$$

in B-mod, where Q lies in $\mathrm{add}(\mathscr{P}(B)_{\mathscr{I}})$ and Q' belongs to B-prinj such that $F(\bar{Y} \oplus I) \oplus Q \simeq F'(\bar{Y} \oplus P) \oplus J$ for some $J \in \mathrm{add}(\mathscr{I}(B)_{\mathscr{P}})$ and $\underline{g'} = F(\underline{g})$ in B-$\underline{\mathrm{mod}}$.

Proof. We provide a proof by using some idea in [9, Lemma 4.13]. Consider the finitely presented functor H:

$$\underline{\mathrm{Hom}}_A(-, Y \oplus \bar{Y} \oplus I \oplus P \oplus P') \longrightarrow \underline{\mathrm{Hom}}_A(-, Z) \longrightarrow H \longrightarrow 0$$

induced from the given minimal exact sequence

$$0 \longrightarrow X \oplus X' \longrightarrow Y \oplus \bar{Y} \oplus I \oplus P \oplus P' \xrightarrow{g} Z \longrightarrow 0$$

in A-mod with $I \in \mathrm{add}(\mathscr{I}(A)_{\mathscr{P}})$, $P \in \mathrm{add}(\mathscr{P}(A)_{\mathscr{I}})$ and $P' \in A$-prinj. It follows from Lemma 2.3 that the sequence of functors

$$\underline{\mathrm{Hom}}_A(-, Y \oplus \bar{Y} \oplus I) \xrightarrow{\mathrm{Hom}_A(-,g)} \underline{\mathrm{Hom}}_A(-, Z) \longrightarrow H \longrightarrow 0$$

is a minimal projective presentation of H in $(A\text{-}\underline{\mathrm{mod}})$-mod and that the sequence of functors

$$0 \longrightarrow H \longrightarrow \mathrm{Ext}_A^1(-, X \oplus X') \longrightarrow \mathrm{Ext}_A^1(-, Y \oplus \bar{Y} \oplus P)$$

is a minimal injective copresentation of H in $(A\text{-}\underline{\mathrm{mod}})$-mod. Applying the equivalence functor α to the above two sequences of functors, we see that the sequence

$$(\star) \qquad \underline{\mathrm{Hom}}_B(-, F(Y \oplus \bar{Y} \oplus I)) \longrightarrow \underline{\mathrm{Hom}}_B(-, F(Z)) \longrightarrow \alpha(H) \longrightarrow 0$$

is a minimal projective presentation of $\alpha(H)$ in $(B\text{-}\underline{\mathrm{mod}})$-mod and the sequence

$$(*) \qquad 0 \longrightarrow \alpha(H) \longrightarrow \mathrm{Ext}_B^1(-, F'(X) \oplus F'(X')) \longrightarrow \mathrm{Ext}_B^1(-, F'(Y \oplus \bar{Y} \oplus P))$$

is a minimal injective copresentation of $\alpha(H)$ in $(B\text{-}\underline{\mathrm{mod}})$-mod.

It follows from $(\star)$ and Lemma 2.4 that there is a minimal exact sequence

$$(\diamondsuit) \qquad 0 \longrightarrow W \longrightarrow F(Y \oplus \bar{Y} \oplus I) \oplus Q \oplus Q' \xrightarrow{g'} F(Z) \longrightarrow 0$$

in B-mod with $Q \in \mathrm{add}(\mathscr{P}(B)_{\mathscr{I}})$, $Q' \in B$-prinj and $\underline{g'} = F(\underline{g})$ in B-$\underline{\mathrm{mod}}$. The minimality of this sequence implies $W \in B\text{-mod}_{\mathscr{I}}$. Note that $(\star)$ is induced from $(\diamondsuit)$. Now, by Lemma 2.3(2) and $(\diamondsuit)$, the exact sequence

$$(\dagger) \qquad 0 \longrightarrow \alpha(H) \longrightarrow \mathrm{Ext}_B^1(-, W) \longrightarrow \mathrm{Ext}_B^1(-, F(Y \oplus \bar{Y} \oplus I) \oplus Q)$$

of functors is a minimal injective copresentation of $\alpha(H)$ in $(B\text{-}\underline{\mathrm{mod}})$-mod. Thus both $(\dagger)$ and $(*)$ are minimal injective copresentations of $\alpha(H)$. This implies that

$$(**) \qquad \mathrm{Ext}_B^1(-, F'(X) \oplus F'(X')) \simeq \mathrm{Ext}_B^1(-, W) \quad \text{and}$$

$$(\ddagger) \qquad \mathrm{Ext}_B^1(-, F'(Y \oplus \bar{Y} \oplus P)) \simeq \mathrm{Ext}_B^1(-, F(Y \oplus \bar{Y} \oplus I) \oplus Q)$$

in $(B\text{-}\underline{\mathrm{mod}})$-mod. Since X lies in $\mathrm{add}(\triangle_A^c)$ and X' lies in $\mathrm{add}(\triangle_A)$, we know from Lemma 2.2 that $F'(X) \in \mathrm{add}(\triangle_B^c)$ and $F'(X') \in \mathrm{add}(\triangle_B)$. In particular, $F'(X) \oplus F'(X') \in B\text{-mod}_{\mathscr{I}}$. Thus $F'(X) \oplus F'(X') \simeq W$ as

B-modules by Lemma 2.1 and ($\ast\ast$). It follows from $X \in \mathrm{add}(\triangle_A^c)$ that $F(X) \simeq F'(X)$ and therefore $F(X) \oplus F'(X') \simeq W$ as B-modules. Hence ($\Diamond$) can be written as

$$0 \longrightarrow F(X) \oplus F'(X') \longrightarrow F(Y \oplus \bar{Y} \oplus I) \oplus Q \oplus Q' \xrightarrow{g'} F(Z) \longrightarrow 0.$$

To complete the proof, we have to show that $F(\bar{Y} \oplus I) \oplus Q \simeq F'(\bar{Y} \oplus P) \oplus J$ for some $J \in \mathrm{add}(\mathscr{I}(B)_{\mathscr{P}})$. In fact, it follows from $Y \in \mathrm{add}(\triangle_A^c)$ that $F(Y) \simeq F'(Y)$ as B-modules and that both $F(Y)$ and $F'(Y)$ lie in B-mod$_{\mathscr{I}}$. Since $\bar{Y}$ belongs to $\mathrm{add}(\mathfrak{n}_F(A))$ and I belongs to $\mathrm{add}(\mathscr{I}(A)_{\mathscr{P}})$, it follows from Lemma 2.2 that $F(\bar{Y} \oplus I)$ lies in $\mathrm{add}(\triangledown_B)$. Thus $F(\bar{Y} \oplus I) \simeq F(\bar{Y}) \oplus F(I) = V \oplus J$ for some $V \in \mathrm{add}(\mathfrak{n}_G(B))$ and $J \in \mathscr{I}(B)_{\mathscr{P}}$. Therefore we have the isomorphisms in (B-mod)-mod:

$$\begin{aligned} \mathrm{Ext}_B^1(-, F(Y \oplus \bar{Y} \oplus I) \oplus Q) &= \mathrm{Ext}_B^1(-, F(Y) \oplus F(\bar{Y} \oplus I) \oplus Q) \\ &\simeq \mathrm{Ext}_B^1(-, F(Y) \oplus V \oplus Q) \\ &\simeq \mathrm{Ext}_B^1(-, F'(Y \oplus \bar{Y} \oplus P)) \quad (\text{ by } (\ddagger)). \end{aligned}$$

As $\bar{Y} \in \mathrm{add}(\mathfrak{n}_F(A))$ and $P \in \mathrm{add}(\mathscr{P}(A)_{\mathscr{I}})$, it follows from Lemma 2.2 that $F'(\bar{Y} \oplus P)$ is in $\mathrm{add}(\triangle_B)$ and $F'(\bar{Y} \oplus P)$ is in B-mod$_{\mathscr{I}}$. Now, Lemma 2.1 shows that $F'(Y \oplus \bar{Y} \oplus P) \simeq F(Y) \oplus V \oplus Q$ and $F'(\bar{Y}) \oplus F'(P) \simeq V \oplus Q$ as B-modules. Thus $F(\bar{Y}) \oplus F(I) \oplus Q \simeq V \oplus J \oplus Q \simeq F'(\bar{Y}) \oplus F'(P) \oplus J$ as B-modules. $\square$

The following special case of Lemma 2.5 is often used in our proofs.

Corollary 2.6. *Let* $0 \to X \oplus X' \to P' \xrightarrow{g} Z \to 0$ *be a minimal exact sequence in A-mod such that* $X \in \mathrm{add}(\triangle_A^c)$, $X' \in \mathrm{add}(\triangle_A)$, $P' \in A$-prinj *and* $Z \in A$-mod$_{\mathscr{P}}$. *Then there exists a minimal exact sequence*

$$0 \longrightarrow F(X) \oplus F'(X') \longrightarrow Q' \xrightarrow{g'} F(Z) \longrightarrow 0$$

of B-modules with $Q' \in B$-prinj.

3 Proofs of the statements

In this section, we prove all results mentioned in Introduction. We keep the notation introduced in the previous sections.

Let A be an Artin algebra over a commutative Artin ring k. Following [17], a projective A-module P is said to be ν-*stably projective* if $\nu_A^i P$ is projective for all $i > 0$. Here ν_A is the Nakayama functor $D\mathrm{Hom}_A(-, A)$ of A. Let U be the direct sum of all non-isomorphic indecomposable ν-stably projective A-modules. Since $\nu_A(U)$ is ν-stably projective, we have $U \simeq \nu_A(U)$ and $\mathrm{top}(U) \simeq \mathrm{soc}(U)$, where $\mathrm{soc}(U)$ is the socle of the A-module U. Clearly, $\mathrm{soc}(U) \simeq \Omega_A(U/\mathrm{soc}(U)) \oplus Q$ for some projective A-module Q. Thus $\mathrm{del}(\mathrm{top}(U)) = \mathrm{del}(\mathrm{soc}(U)) = 0$ by definition. Let V be the direct sum of all non-isomorphic indecomposable projective A-modules that are neither simple nor ν-stably projective. Then $\mathrm{del}(A) = \mathrm{del}(\mathrm{top}(U \oplus V)) = \max\{\mathrm{del}(\mathrm{top}(U)), \mathrm{del}(\mathrm{top}(V))\} = \mathrm{del}(\mathrm{top}(V))$.

3.1 Proof of Theorem 1.1

Let A and B be Artin k-algebras that have neither nodes nor semisimple direct summands. Assume that $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ is an equivalence of k-categories.

Under these assumptions, $\mathfrak{n}_F(A) = \varnothing$, $\mathfrak{n}_{F^{-1}}(B) = \varnothing$ and there is a bijection $F' : \mathscr{P}(A)_{\mathscr{I}} \to \mathscr{P}(B)_{\mathscr{I}}$ (see Lemma 2.2). Further, Lemma 2.5 can be specified as follows.

Lemma 3.1. [26, Theorem 1.7] *Let* $0 \to X \oplus P_1 \xrightarrow{f} Y \oplus P \oplus P' \xrightarrow{g} Z \to 0$ *be a minimal exact sequence of A-modules, where* $X, Y, Z \in A$-mod$_{\mathscr{P}}$, $P_1, P \in \mathscr{P}(A)_{\mathscr{I}}$ *and P' is a projective-injective A-module. Then there is a minimal exact sequence*

$$0 \longrightarrow F(X) \oplus F'(P_1) \xrightarrow{f'} F(Y) \oplus F'(P) \oplus Q \xrightarrow{g'} F(Z) \longrightarrow 0$$

in B-mod *with Q a projective-injective B-module and $\underline{g}' = F(\underline{g})$. In particular, $\Omega_B F(Z) \simeq F\Omega_A(Z)$ in B*-$\underline{\text{mod}}$ *for $Z \in A$-mod$_{\mathscr{P}}$.*

Lemma 3.2. *For $X \in A$-mod$_{\mathscr{P}}$, we have $\text{del}(X) = \text{del}(F(X))$.*

Proof. We show $\text{del}(F(X)) \leq \text{del}(X)$. In fact, we may assume $d := \text{del}(X) < \infty$. Then, by the definition of delooping levels, there exists $M \in A$-mod$_{\mathscr{P}}$ such that $\Omega_A^d(X) \in \text{add}(_A A \oplus \Omega_A^{d+1}(M))$. Thus $F(\Omega_A^d(X)) \in \text{add}(_B B \oplus F\Omega_A^{d+1}(M))$ by the additivity of the functor F. On the other hand, it follows from Lemma 3.1 that $\Omega_B F(X) \simeq F\Omega_A(X)$ and $\Omega_B F(M) \simeq F\Omega_A(M)$ in B-$\underline{\text{mod}}$. Then $\Omega_B^i F(X) \simeq F\Omega_A^i(X)$ and $\Omega_B^i F(M) \simeq F\Omega_A^i(M)$ in B-$\underline{\text{mod}}$ for $i \geq 0$. Thus $\Omega_B^d F(X) \in \text{add}(_B B \oplus \Omega_A^{d+1} F(M))$, and therefore $\text{del}(F(X)) \leq d = \text{del}(X) < \infty$. Similarly, we show $\text{del}(X) \leq \text{del}(F(X))$. Thus $\text{del}(X) = \text{del}(F(X))$. $\square$

Proof of Theorem 1.1. Suppose that A and B are stably equivalent Artin algebras without nodes.

(i) $\text{del}(A) = \text{del}(B)$. In fact, the delooping levels of algebras involve only simple modules. If A or B has a semisimple direct summand, then the simple modules belonging to the semisimple direct summand have delooping levels 0, and therefore do not contribute to the delooping levels of the considered algebra. So we may remove all semisimple direct summands from both algebras A and B. Of course, the resulting algebras are still stably equivalent. Thus we assume that both A and B do not have any semisimple direct summands. Let V be the direct sum of all non-isomorphic indecomposable projective A-modules that are neither simple nor ν-stably projective, and let V' be the direct sum of all non-isomorphic indecomposable projective B-modules which are neither simple nor ν-stably projective. It follows from [26, Lemma 2.5], which is true also for Artin algebras, that $F(\text{top}(V)) \simeq \text{top}(V')$ as B-modules. Note that $\text{top}(V)$ does not have any nonzero projective direct summands. By Lemma 3.2, we have $\text{del}(\text{top}(V)) = \text{del}(\text{top}(V'))$. Thus $\text{del}(A) = \text{del}(\text{top}(V)) = \text{del}(\text{top}(V')) = \text{del}(B)$.

(ii) $\phi\dim(A) = \phi\dim(B)$ and $\psi\dim(A) = \psi\dim(B)$. For Artin algebras A_1 and A_2, there hold $\phi\dim(A_1 \times A_2) = \max\{\phi\dim(A_1), \phi\dim(A_2)\}$ and $\psi\dim(A_1 \times A_2) = \max\{\psi\dim(A_1), \psi\dim(A_2)\}$. Since ϕ- and ψ-dimensions of semisimple algebras are 0, we may remove semisimple direct summands from A and B if they have any. Then the resulting algebras are still stably equivalent. So we may assume that both algebras A and B do not have any semisimple direct summands. Let $F : A$-$\underline{\text{mod}} \to B$-$\underline{\text{mod}}$ defines a stable equivalence between A and B. We also denote by F the correspondence from A-modules to B-modules, which takes projective A-module to 0. As a functor of k-categories, F is additive and commutes with finite direct sums in A-$\underline{\text{mod}}$. Thus the map $\tilde{F} : K(A) \to K(B)$ given by $\tilde{F}([X]) := [F(X)]$, is a well-defined homomorphism of Grothendieck groups. It is actually an isomorphism of abelian groups. By Lemma 3.1 we have $\Omega_B(F(X)) \simeq F(\Omega_A(X))$ in B-$\underline{\text{mod}}$ for $X \in A$-mod. Let $\langle X \rangle$ be the $\mathbb{Z}$-submodule of $K(A)$ generated by the isomorphism classes of indecomposable, non-projective direct summands of X. For $n \geq 0$, the following diagrams are commutative

$$
\begin{array}{ccc}
K(A) & \xrightarrow{\tilde{F}} & K(B) \\
\downarrow{\scriptstyle\Omega_A} & & \downarrow{\scriptstyle\Omega_B} \\
K(A) & \xrightarrow{\tilde{F}} & K(B),
\end{array}
\qquad
\begin{array}{ccc}
\Omega_A^n \langle X \rangle & \xrightarrow{\tilde{F}_{res}} & \Omega_B^n \langle F(X) \rangle \\
\downarrow{\scriptstyle\Omega_A} & & \downarrow{\scriptstyle\Omega_B} \\
\Omega_A^{n+1} \langle X \rangle & \xrightarrow{\tilde{F}_{res}} & \Omega_B^{n+1} \langle F(X) \rangle
\end{array}
$$

where $\tilde{F}_{res}$ is the restriction of $\tilde{F}$. Since $\tilde{F} : K(A) \to K(B)$ is an isomorphism of abelian groups, the $\mathbb{Z}$-module homomorphism $\Omega_A : \Omega_A^n \langle X \rangle \to \Omega_A^{n+1} \langle X \rangle$ is isomorphic for $n \geq 0$ if and only if so is the $\mathbb{Z}$-homomorphism $\Omega_B : \Omega_B^n \langle F(X) \rangle \to \Omega_B^{n+1} \langle F(X) \rangle$ for $n \geq 0$. By the definition of ϕ-dimensions, $\phi(X) = \phi(F(X))$ and $\phi\dim(A) \leq \phi\dim(B)$. Similarly, $\phi\dim(B) \leq \phi\dim(A)$. Thus $\phi\dim(A) = \phi\dim(B)$. For $Y \in A$-mod, since $\Omega_B(F(Y)) \simeq F(\Omega_A(Y))$ in B-$\underline{\text{mod}}$ and F is an equivalence, we get $\text{projdim}(_B F(Y)) = \text{projdim}(_A Y)$. Then $\psi(X) = \psi(F(X))$ and $\psi\dim(A) = \psi\dim(B)$. $\square$

Theorem 1.1 may fail if Artin algebras have nodes. This can be seen by the following examples.

Example 3.3. (1) Let A_1 be the algebra over a field k, given by the quiver with a relation:

$$1\bullet \circlearrowleft \alpha\,, \qquad \alpha^2 = 0.$$

Clearly, A_1 has a node and is stably equivalent to the path algebra A_1' of the quiver $1\bullet \leftarrow \bullet 2$. Note that A_1' has no nodes and its Frobenius part is 0. In this case, both A_1 and A_1' have only 1 non-projective simple module. Clearly, $\mathrm{del}(A_1) = \phi\dim(A_1) = \psi\dim(A_1) = 0 < 1 = \mathrm{del}(A_1') = \phi\dim(A_1') = \psi\dim(A_1')$. Remark that A_1 and A_1' are never stably equivalent of Morita type by Lemma 3.9 below.

(2) Let A_2 be the algebra over a field k, given by the quiver with a relation:

$$1\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet 2\,, \qquad \alpha\beta = 0$$

(see [32, Example 4.9] for more general situations). In this case, we consider the 2 almost split sequences in A_2-mod

$$0 \longrightarrow S(1) \longrightarrow P(2) \longrightarrow S(2) \longrightarrow 0 \ \text{ and } \ 0 \longrightarrow S(2) \longrightarrow I(2) \longrightarrow S(1) \longrightarrow 0,$$

where $P(i)$, $I(i)$ and $S(i)$ are the projective, injective and simple modules corresponding to the vertex i, respectively. Clearly, A_2 has the Frobenius part isomorphic to A_1, and a unique node $S(1)$. Let I be the trace of $S(1)$ in A_2 and J be the left annihilator of I in A_2. Then $I = \{r_1\beta\alpha + r_2\beta \mid r_1, r_2 \in k\}$ and $J = \{r_1 e_2 + r_2\alpha + r_3\beta \mid r_i \in k, 1 \le i \le 3\}$. Define A_2' to be the triangular matrix algebra

$$A_2' = \begin{pmatrix} A_2/I & 0 \\ I & A_2/J \end{pmatrix} \simeq \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix}.$$

Then A_2' has no nodes and its Frobenius part is 0. By Lemma 3.13(2), A_2 and A_2' are stably equivalent. Particularly, they have 2 non-isomorphic, non-projective simple modules, and $\mathrm{del}(A_2) = \phi\dim(A_2) = \psi\dim(A_2) = 2 > 1 = \mathrm{del}(A_2') = \phi\dim(A_2') = \psi\dim(A_2')$.

Neither delooping levels nor ϕ-dimensions are preserved by tilting. For example, the path algebra A (over a field) of the quiver $\bullet \overset{\alpha}{\longrightarrow} \bullet \overset{\beta}{\longrightarrow} \bullet$ can be tilted to the quotient algebra $B := A/(\alpha\beta)$. In this case, A has no nodes, but B has a node, while we have $\mathrm{del}(A) = \phi\dim(A) = \psi\dim(A) = 1 < 2 = \mathrm{del}(B) = \phi\dim(B) = \psi\dim(B)$. This shows that in general derived equivalences do not have to preserve the delooping levels and the ϕ-dimensions of algebras. Nevertheless, we will show that almost ν-stable derived equivalences do preserve delooping levels, ϕ-dimensions and ψ-dimensions.

Proposition 3.4. *Let A and B be arbitrary finite-dimensional algebras over a field. If A and B are almost ν-stable derived equivalent, then $\mathrm{del}(A) = \mathrm{del}(B)$, $\phi\dim(A) = \phi\dim(B)$ and $\psi\dim(A) = \psi\dim(B)$. In particular, if A is a self-injective algebra over a field and $X \in A$-mod, then*
(1) $\mathrm{del}\big(\mathrm{End}_A(A\oplus X)\big) = \mathrm{del}\big(\mathrm{End}_A(A\oplus\Omega_A(X))\big) = \mathrm{del}\big(\mathrm{End}_A(A\oplus D\mathrm{Tr}(X))\big)$.
(2) $\phi\dim\big(\mathrm{End}_A(A\oplus X)\big) = \phi\dim\big(\mathrm{End}_A(A\oplus\Omega_A(X))\big) = \phi\dim\big(\mathrm{End}_A(A\oplus D\mathrm{Tr}(X))\big)$.
(3) $\psi\dim\big(\mathrm{End}_A(A\oplus X)\big) = \psi\dim\big(\mathrm{End}_A(A\oplus\Omega_A(X))\big) = \psi\dim\big(\mathrm{End}_A(A\oplus D\mathrm{Tr}(X))\big)$.

Before starting with the proof of this proposition, we recall two definitions.

Definition 3.5. (see [16]) *A derived equivalence F of bounded derived module categories between arbitrary Artin algebras A and B with a quasi-inverse G is said to be* almost ν-stable *if the associated radical tilting complexes $T^\bullet$ over A to F and $\bar{T}^\bullet$ over B to G are of the form*

$$T^\bullet : 0 \longrightarrow T^{-n} \longrightarrow \cdots T^{-1} \longrightarrow T^0 \longrightarrow 0 \ \text{ and } \ \bar{T}^\bullet : 0 \longrightarrow \bar{T}^0 \longrightarrow \bar{T}^1 \longrightarrow \cdots \longrightarrow \bar{T}^n \longrightarrow 0,$$

respectively, such that $\mathrm{add}(\bigoplus_{i=1}^n T^{-i}) = \mathrm{add}(\bigoplus_{i=1}^n \nu_A(T^{-i}))$ and $\mathrm{add}(\bigoplus_{i=1}^n \bar{T}^i) = \mathrm{add}(\bigoplus_{i=1}^n \nu_B(\bar{T}^i))$.

Almost ν-stable derived equivalences induce special stable equivalences (see [16, Theorem 1.1(2)]), namely stable equivalences of Morita type.

Definition 3.6. *Let A and B be arbitrary finite-dimensional algebras over a field k.*

(1) *A and B are said to be* stably equivalent of Morita type *(see [8]) if there exist bimodules $_AM_B$ and $_BN_A$ such that*

(i) *M and N are projective as one-sided modules,*

(ii) *$M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.*

(2) *A and B are said to be* stably equivalent of adjoint type *(see [31]) if the bimodules M and N in* (1) *provide additionally two adjoint pairs $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$ of functors on module categories.*

Stable equivalences of adjoint type have some nice properties.

Lemma 3.7. *Let A and B be arbitrary finite-dimensional algebras over a field. Suppose A and B are stably equivalent of adjoint type induced by $_AM_B$ and $_BN_A$. Write $_AM \otimes_B N_A \simeq A \oplus P$ and $_BN \otimes_A M_B \simeq B \oplus Q$ as bimodules. Then the following hold:*

(1) $\mathrm{add}(\nu_A P) = \mathrm{add}(_AP)$ *and* $\mathrm{add}(\nu_B Q) = \mathrm{add}(_BQ)$, *where ν_A is the Nakayama functor of A.*

(2) *If S is a simple A-module with $\mathrm{Hom}_A(P, S) = 0$, then $N \otimes_A S$ is a simple B-module with $\mathrm{Hom}_B(Q, N \otimes_A S) = 0$.*

(3) *For an A-module X and a B-module Y, we have $\Omega_B^i(N \otimes_A X) \simeq N \otimes_A \Omega_A^i(X)$ in B-$\underline{\mathrm{mod}}$ and $\Omega_A^i(M \otimes_B Y) \simeq M \otimes_B \Omega_B^i(Y)$ in A-$\underline{\mathrm{mod}}$ for $i \geq 0$.*

(4) *For an A-module X and a B-module Y, there hold $\mathrm{del}(_AM \otimes_B N \otimes_A X) = \mathrm{del}(_BN \otimes_A X) = \mathrm{del}(_AX)$ and $\mathrm{del}(_BN \otimes_A M \otimes_B Y) = \mathrm{del}(_AM \otimes_B Y) = \mathrm{del}(_BY)$.*

Proof. (1) and (2) follow from [17, Lemma 3.1], while (3) and (4) can be deduced easily. □

Lemma 3.8. *Let A and B be arbitrary Artin algebras with no separable direct summands. If A and B are stably equivalent of Morita type, then they are even stably equivalent of adjoint type.*

Proof. Since A and B are stably equivalent of Morita type and have no separable direct summands, it follows from [22, Proposition 2.1 and Theorem 2.2] which are valid also for Artin algebras by checking the argument there, that A and B have the same number of indecomposable direct summands (as two-sided ideals) and that we may write $A = A_1 \times A_2 \times \cdots \times A_s$ and $B = B_1 \times B_2 \times \cdots \times B_s$ as products of indecomposable algebras, such that the blocks A_i and B_i are stably equivalent of Morita type for $1 \leq i \leq s$. Suppose that $M^{(i)}$ and $N^{(i)}$ define a stable equivalence of Morita type between A_i and B_i. Observe that the two results [11, Lemma 2.1 and Corollary 3.1] hold true for indecomposable and non-separable Artin algebras. Thus, by [11, Lemma 2.1], we may assume that $_{A_i}M^{(i)}{}_{B_i}$ and $_{B_i}N^{(i)}{}_{A_i}$ are indecomposable and non-projective bimodules. Since the algebra A_i is indecomposable and non-separable by assumption, it follows from [11, Corollary 3.1] that $(M^{(i)} \otimes_{B_i} -, N^{(i)} \otimes_{A_i} -)$ and $(N^{(i)} \otimes_{A_i} -, M^{(i)} \otimes_{B_i} -)$ are adjoint pairs between A_i-mod and B_i-mod. Let $M := \bigoplus_{1 \leq j \leq s} M^{(j)}$ and $N := \bigoplus_{1 \leq j \leq s} N^{(j)}$. Then $_AM_B$ and $_BN_A$ define a stable equivalence of adjoint type between A and B. □

For stable equivalences of Morita type, the requirement that algebras considered have no nodes can be eliminated.

Lemma 3.9. *Let A and B be arbitrary finite-dimensional algebras over a field k. If A and B are stably equivalent of Morita type, then*

(1) $\mathrm{del}(A) = \mathrm{del}(B)$.

(2) $\phi\dim(A) = \phi\dim(B)$ *and* $\psi\dim(A) = \psi\dim(B)$.

Proof. (1) Let $A = A_0 \times A_1$ and $B = B_0 \times B_1$, where A_0 and B_0 are separable algebras, and where A_1 and B_1 are algebras without separable direct summands. Since A and B are stably equivalent of Morita type, it follows from the proof of [23, Theorem 4.7] that A_1 and B_1 are stably equivalent of Morita type. By Lemma 3.8, A_1 and B_1 are stably equivalent of adjoint type. Suppose that the adjoint type between A_1 and B_1 is defined by two bimodules ${}_{A_1}M_{B_1}$ and ${}_{B_1}N_{A_1}$. By Definition 3.6, we write ${}_{A_1}M \otimes_{B_1} N_{A_1} \simeq A_1 \oplus P$ and ${}_{B_1}N \otimes_{A_1} M_{B_1} \simeq B_1 \oplus Q$. It follows from Lemma 3.7(1) that $\mathrm{add}(\nu_{A_1}P) = \mathrm{add}({}_{A_1}P)$. Then $\mathrm{del}(\mathrm{top}(P)) = \mathrm{del}(\mathrm{soc}(P)) = 0$ and

$$\mathrm{del}(A_1) = \max\{\mathrm{del}({}_{A_1}S) \mid {}_{A_1}S \text{ is simple with } \mathrm{Hom}_{A_1}(P,S) = 0\}.$$

Let S be a simple A_1-module with $\mathrm{Hom}_{A_1}(P,S) = 0$. By Lemma 3.7(2), ${}_{B_1}N \otimes_{A_1} S$ is a simple B_1-module. It follows from Lemma 3.7(4) that $\mathrm{del}({}_{A_1}S) = \mathrm{del}({}_{B_1}N \otimes_{A_1} S) \leq \mathrm{del}(B_1)$, and therefore $\mathrm{del}(A_1) \leq \mathrm{del}(B_1)$. Similarly, we prove $\mathrm{del}(B_1) \leq \mathrm{del}(A_1)$. Thus $\mathrm{del}(A_1) = \mathrm{del}(B_1)$. Since the delooping levels of separable blocks are 0, we have $\mathrm{del}(A) = \max\{\mathrm{del}(A_0), \mathrm{del}(A_1)\} = \mathrm{del}(A_1) = \mathrm{del}(B_1) = \max\{\mathrm{del}(B_0), \mathrm{del}(B_1)\} = \mathrm{del}(B)$.

(2) Suppose that A and B are stably equivalent of Morita type defined by ${}_AM_B$ and ${}_BN_A$. Since ${}_BN$ is projective, the functor $F := N \otimes_A - : A\text{-mod} \to B\text{-mod}$ takes projective A-modules to projective B-modules, and commutes with finite direct sums. Thus F induces an equivalence: $A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$. As in the proof of Theorem 1.1(ii), we obtain $\phi\dim(A) = \phi\dim(B)$ and $\psi\dim(A) = \psi\dim(B)$. $\square$

Proof of Proposition 3.4. Suppose that there is an almost ν-stable derived equivalence between finite-dimensional algebras A and B over a field k. It follows from [16, Theorem 1.1(2)] that A and B are stably equivalent of Morita type. By Lemma 3.9, $\mathrm{del}(A) = \mathrm{del}(B)$, $\phi\dim(A) = \phi\dim(B)$ and $\psi\dim(A) = \psi\dim(B)$.

Let A be a self-injective algebra over a field and $X \in A\text{-mod}$. Then $\mathrm{End}_A(A \oplus X)$ and $\mathrm{End}_A(A \oplus \Omega_A(X))$ are almost ν-stable derived equivalent by [18, Corollary 3.14] (see also the remark at the end of Section 3 in [16]), and therefore they are stably equivalent of Morita type. Hence

$$(\sharp) \quad \mathrm{del}\big(\mathrm{End}_A(A \oplus X)\big) = \mathrm{del}\big(\mathrm{End}_A(A \oplus \Omega_A(X))\big).$$

As ν_A is an auto-equivalence of $A\text{-mod}$ and $D\mathrm{Tr}(Y) \simeq \Omega^2(\nu_A(Y))$ in $A\text{-}\underline{\mathrm{mod}}$ for $Y \in A\text{-mod}$, we have

$$\begin{aligned}
\mathrm{del}\big(\mathrm{End}_A(A \oplus X)\big) &= \mathrm{del}\big(\mathrm{End}_A(\nu_A(A \oplus X))\big) = \mathrm{del}\big(\mathrm{End}_A(A \oplus \nu_A(X))\big) \\
&= \mathrm{del}\big(\mathrm{End}_A(A \oplus \Omega(\nu_A(X)))\big) \ (\text{ by } (\sharp)) \\
&= \mathrm{del}\big(\mathrm{End}_A(A \oplus \Omega^2(\nu_A(X)))\big) \ (\text{ by } (\sharp)) \\
&= \mathrm{del}\big(\mathrm{End}_A(A \oplus D\mathrm{Tr}(X))\big),
\end{aligned}$$

where the last equality is due to the fact that $\mathrm{End}_A(A \oplus \Omega^2(\nu_A(X)))$ and $\mathrm{End}_A(A \oplus D\mathrm{Tr}(X))$ are Morita equivalent. Similarly, we can prove the equalities for ϕ-dimensions and ψ-dimensions by Lemma 3.9(2). $\square$

Proposition 3.4 can distinguish almost ν-stable derived equivalences out of derived equivalences. For instance, let A and B be algebras given by the following quivers Q_A and Q_B with relations, respectively:

$$Q_A: \quad \overset{1}{\bullet} \underset{\delta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet} \underset{\gamma}{\overset{\beta}{\rightleftarrows}} \overset{3}{\bullet} \qquad\qquad Q_B: \quad \overset{1}{\bullet} \overset{\alpha'}{\longrightarrow} \overset{2}{\bullet}$$

$$\alpha\delta\alpha = \gamma\delta = \delta\alpha - \beta\gamma = 0 \,; \qquad\qquad \alpha'\beta'\gamma'\alpha' = \gamma'\alpha'\beta'\gamma' = 0.$$

It was shown in [19, Example 4.10] that A and B are derived equivalent. One can check that both algebras have no nodes and $\mathrm{del}(A) = 2 \neq 1 = \mathrm{del}(B)$. Thus A and B are neither almost ν-stable derived equivalent by Proposition 3.4 nor stably equivalent by Theorem 1.1.

3.2 Proof of Theorem 1.3

Now we turn to the proof of Theorem 1.3. We first recall the notion of ν-dominant dimensions.

Let A be an Artin algebra. We denote by A-prinj the full subcategory of A-mod consisting of those A-modules that are both projective and injective. For an A-module $M \in A$-mod, we consider its minimal injective resolution

$$0 \longrightarrow {}_A M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots.$$

Let I be an injective A-module and $0 \leq n \leq \infty$. If n is maximal such that all modules I_j are in $\mathrm{add}(I)$ for $j < n$, then n is called the *I-dominant dimension* of M, denoted by I-dom.dim(M). Dually, we consider its minimal projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_A M \longrightarrow 0.$$

Let P be a projective A-module and $0 \leq m \leq \infty$. If m is maximal such that all modules P_j are in $\mathrm{add}(P)$ for $j < m$, then m is called the *P-codominant dimension* of M, denoted by P-codom.dim(M). Clearly, the codominant dimension of M and the dominant dimension of A^{op}-module $D(M)$ are equal. Now, if $\mathrm{add}(I) = \mathrm{add}(P) = A$-prinj, then we define the *dominant dimension* of M to be I-dom.dim(M), denoted by dom.dim(M); the *codominant dimension* of M to be P-codom.dim(M), denoted by codom.dim(M), and the *dominant dimension* of the algebra A to be dom.dim$({}_A A)$, denoted by dom.dim(A). Note that dom.dim$(A) = $ dom.dim(A^{op}) (see [27, Theorem 4] or [15]). It is clear that dom.dim$(A) = \min\{\mathrm{dom.dim}(P) \mid P \in \mathrm{add}({}_A A)\}$. If $\mathrm{add}(I) = A$-stp, then I-dom.dim(M) is called the ν-*dominant dimension* of M, denoted by ν-dom.dim(M). The ν-*dominant dimension of the algebra A* is defined to be ν-dom.dim$({}_A A)$.

Lemma 3.10. *Let A be an Artin algebra with ν-dom.dim$(A) \geq 1$. Then*

(1) *A-stp $= A$-prinj and ν-dom.dim$(A) = $ dom.dim(A).*

(2) *The projective cover of a simple module ${}_A S$ is injective if and only if the injective envelope of S is projective.*

(3) *If the projective cover of a simple module ${}_A S$ is not injective, then S itself is neither projective nor injective.*

Proof. (1) and (2) are trivial. We prove (3). Let P and I be the projective cover and injective envelope of S, respectively. By assumption, $P \notin A$-prinj. By (2), $I \notin A$-prinj. If S is injective, then it follows from codom.dim$(D(A_A)) = $ dom.dim$(A) = \nu$-dom.dim$(A) \geq 1$ that codom.dim$(S) \geq 1$. This implies $P \in A$-prinj, a contradiction. Thus S is not an injective module. Suppose that S is projective. It follows from dom.dim$(A) \geq 1$ that dom.dim$(S) \geq 1$ and $I \in A$-prinj, again a contradiction. Thus S is not a projective module. $\square$

Lemma 3.11. *Let $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ define a stable equivalence between Artin algebras A and B, and let G be a quasi-inverse of F. If ν-dom.dim$(A) \geq 1$ and ν-dom.dim$(B) \geq 1$, then there exist bijections*

$$F : \mathscr{I}(A)_{\mathscr{P}} \longrightarrow \mathscr{I}(B)_{\mathscr{P}}, \ F : \mathfrak{n}_F(A) \longrightarrow \mathfrak{n}_G(B), \ F' : \mathscr{P}(A)_{\mathscr{I}} \longrightarrow \mathscr{P}(B)_{\mathscr{I}} \text{ and } F' : \mathfrak{n}_F(A) \longrightarrow \mathfrak{n}_G(B).$$

Proof. Suppose $I \in \mathscr{I}(A)_{\mathscr{P}}$, we show $F(I) \in \mathscr{I}(B)_{\mathscr{P}}$. Indeed, let S be the socle of I. By Lemma 3.10(2)-(3), S is not injective. Thus $S \not\simeq I$ and the natural projection $\pi : I \to I/S$ is an irreducible map. Since I is not a projective module, we have $I/S \in A$-mod$_{\mathscr{P}}$. Thus $0 \neq \pi \in A$-$\underline{\mathrm{mod}}$ and $0 \neq F(\pi) \in B$-$\underline{\mathrm{mod}}$. By [6, Chapter X, Proposition 1.3], $F(\pi) : F(I) \to F(I/S)$ is irreducible. By Lemma 2.2, we have $F(I) \in \bigtriangledown_A$, namely $F(I) \in \mathfrak{n}_G(B)$ or $F(I) \in \mathscr{I}(B)_{\mathscr{P}}$. Suppose $F(I) \in \mathfrak{n}_G(B)$. Then $F(I)$ is a node and there is an almost split sequence $0 \to F(I) \to Q \to \mathrm{Tr}\,D(F(I)) \to 0$ with ${}_B Q$ projective. Since $F(\pi)$ is irreducible and ${}_B F(I)$ is indecomposable, by [6, Chapter V, Theorem 5.3], we get $F(I/S) \in \mathrm{add}({}_B Q)$. Thus $F(I/S)$ is a projective B-module and $F(\pi) = 0$ in B-$\underline{\mathrm{mod}}$. This is a contradiction and shows $F(I) \in \mathscr{I}(B)_{\mathscr{P}}$.

Similarly, we show that $G(J)$ lies in $\mathscr{I}(A)_{\mathscr{P}}$ for $J \in \mathscr{I}(B)_{\mathscr{P}}$. By Lemma 2.2, $F : \mathscr{I}(A)_{\mathscr{P}} \to \mathscr{I}(B)_{\mathscr{P}}$ and $F : \mathfrak{n}_F(A) \to \mathfrak{n}_G(B)$ are bijections.

Let $P \in \mathscr{P}(A)_{\mathscr{I}}$ with S as its top. By Lemma 3.10(3), S is not projective. Now, Lemma 2.5 implies $F'(P) \in B$-proj. Thus $F'(P) \in \triangle_A$ by Lemma 2.2, and therefore $F'(P) \in \mathscr{P}(B)_{\mathscr{I}}$. Similarly, $G'(Q) \in \mathscr{P}(A)_{\mathscr{I}}$ for $Q \in \mathscr{P}(B)_{\mathscr{I}}$. Thus Lemma 2.2 yields the bijections $F' : \mathscr{P}(A)_{\mathscr{I}} \to \mathscr{P}(B)_{\mathscr{I}}$ and $F' : \mathfrak{n}_F(A) \to \mathfrak{n}_G(B)$. $\square$

Before starting the proof of Theorem 1.3, we introduce a few notation.

Suppose that X is an A-module such that A-stp $= \mathrm{add}(_AX)$, where A-stp stands for the full subcategory of A-mod consisting of all ν-stably projective A-modules. By $\mathrm{pre}(X)$ we denote the full subcategory of A-mod consisting of all those A-modules M that have a minimal projective presentation $P_1 \to P_0 \to M \to 0$ with $P_1, P_2 \in \mathrm{add}(X)$. Let $\Lambda := \mathrm{End}_A(X)$. It follows from [6, Chapter II, Proposition 2.5] that the functor $\mathrm{Hom}_A(X,-) : \mathrm{pre}(X) \to \Lambda$-mod is an equivalence of additive categories with a quasi-inverse $X \otimes_\Lambda - : \Lambda$-mod $\to \mathrm{pre}(X)$. Let $R_X(M,N)$ be the k-submodule of $\mathrm{Hom}_A(M,N)$ consisting of all those homomorphisms of A-modules that factorize through a module in $\mathrm{add}(X)$. Then R_X is an ideal of the k-category $\mathrm{pre}(X)$. Note that $R_X(N,M) = R_A(N,M)$ for all $N \in A$-mod and $M \in \mathrm{pre}(X)$ because a homomorphism $f : N \to M$ of A-modules factorizes through a projective A-module must factorize through the projective cover $P_0 \to M$. We denote by $\underline{\mathrm{pre}}(X)$ the quotient category of $\mathrm{pre}(X)$ modulo the ideal R_X. Thus $\underline{\mathrm{pre}}(X)$ is a full additive subcategory of A-$\underline{\mathrm{mod}}$.

Proof of Theorem 1.3. Suppose that $F : A$-$\underline{\mathrm{mod}} \to B$-$\underline{\mathrm{mod}}$ defines a stable equivalence between Artin algebras A and B, where both algebras have positive ν-dominant dimensions. Then A-stp $= A$-prinj and B-stp $= B$-prinj by Lemma 3.10(1). Let $_AX$ and $_BY$ be modules such that A-stp $= \mathrm{add}(_AX)$ and B-stp $= \mathrm{add}(_BY)$, respectively. We define $\Lambda := \mathrm{End}(X)$ and $\Gamma := \mathrm{End}_B(Y)$. Then Λ and Γ are the Frobenius parts of A and B, respectively. Moreover, Λ and Γ are self-injective algebras

To show that Λ and Γ are stably equivalent, it is enough to show that F induces an equivalence from $\underline{\mathrm{pre}}(X)$ to $\underline{\mathrm{pre}}(Y)$. Since F is an equivalence, we need only to show that $F(M)$ lies in $\mathrm{pre}(Y)$ for all $M \in \mathrm{pre}(X)$.

In fact, take $M \in \mathrm{pre}(X)$ and a minimal projective presentation: $P_1 \to P_0 \to M \to 0$ with $P_1, P_0 \in \mathrm{add}(_AX)$. We may assume that M has no nonzero projective direct summands. Then the exact sequence

$$0 \longrightarrow \Omega_A(M) \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is minimal and $\Omega_A(M)$ does not have any injective direct summands, that is, $\Omega_A(M) \in A$-$\mathrm{mod}_{\mathscr{I}}$. Since P_1 is a projective-injective A-module, we have $\Omega_A(M) \in A$-$\mathrm{mod}_{\mathscr{P}}$. So we write $\Omega_A(M) \simeq K_1 \oplus K_2$ with $K_1 \in \mathrm{add}(\triangle_A^c)$ and $K_2 \in \mathrm{add}(\mathfrak{n}_F(A))$. By Corollary 2.6, we have a minimal exact sequence

$$(*) \quad 0 \longrightarrow F(K_1) \oplus F'(K_2) \longrightarrow Q_0 \longrightarrow F(M) \longrightarrow 0$$

in B-mod with $Q_0 \in B$-prinj $= \mathrm{add}(Y)$.

Next, we investigate the projective covers of $F(K_1)$ and $F'(K_2)$. Let P_1' be the projective cover of K_1. Then $P_1' \in \mathrm{add}(_AP_1)$ and $P_1' \in A$-prinj. Note that $\Omega_A(K_1)$ lies in A-$\mathrm{mod}_{\mathscr{I}}$ and we can write $\Omega_A(K_1) = L_1 \oplus L_2$ with $L_1 \in \mathrm{add}(\triangle_A^c)$ and $L_2 \in \mathrm{add}(\Delta_A)$. Applying Corollary 2.6 to the minimal exact sequence

$$0 \longrightarrow \Omega_A(K_1) \longrightarrow P_1' \longrightarrow K_1 \longrightarrow 0$$

in A-mod, we get a minimal exact sequence of B-modules

$$0 \longrightarrow F(K_1)) \oplus F'(L_2) \longrightarrow Q_1' \longrightarrow F(K_1) \longrightarrow 0$$

with $Q_1' \in B$-prinj.

Now, we investigate the projective cover of $F'(K_2)$. By Lemma 3.11, it follows from $K_2 \in \mathrm{add}(\mathfrak{n}_F(A))$ that $F'(K_2) \in \mathrm{add}(\mathfrak{n}_G(B))$, where G is the quasi-inverse of the functor F. From the sequence $(*)$ and $Q_0 \in B$-prinj, we infer that the injective envelope of $F'(K_2)$ is projective. Since nodes are simple modules, it follows from Lemma 3.10(2) that the projective cover $_BQ_2'$ of $F'(K_2)$ is projective-injective. Thus the minimal projective presentation of $F(M)$ is as follows.

$$Q_1' \oplus Q_2' \longrightarrow Q_0 \longrightarrow F(M) \longrightarrow 0$$

with $Q_1', Q_2', Q_0 \in B$-prinj. This yields $F(M) \in \mathrm{pre}(Y)$. Similarly, we prove that the quasi-inverse G of F sends $N \in \mathrm{pre}(Y)$ to $G(N) \in \mathrm{pre}(X)$.

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Finally, we reach to the commutative diagram of functors in stable module categories:

$$\underline{\mathrm{pre}}(X) \underset{G.}{\overset{F.}{\rightleftarrows}} \underline{\mathrm{pre}}(Y)$$

$$A\text{-}\underline{\mathrm{mod}} \underset{G}{\overset{F}{\rightleftarrows}} B\text{-}\underline{\mathrm{mod}}$$

where $F.$ and $G.$ stand for the restrictions of F and G to $\underline{\mathrm{pre}}(X)$ and $\underline{\mathrm{pre}}(Y)$, respectively. It follows from the equivalence of F that $F.$ is an equivalence of k-categories. $\square$

The following example shows that the assumption of ν-dominant dimensions in Theorem 1.3 cannot be dropped.

Example 3.12. Let A and B be algebras given by the quivers with relations:

$$A: \quad 1\bullet \overset{\alpha}{\longrightarrow} \bullet 2 \qquad\qquad B: \quad 1'\bullet \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} \bullet 2' \quad 3'\bullet \underset{\delta'}{\overset{\gamma'}{\rightleftarrows}} \bullet 4'$$

with δ up-arrow on left, β down-arrow on right, $4\bullet \overset{\gamma}{\longleftarrow} \bullet 3$, and $\beta\gamma = \delta\alpha = 0$, while $\beta'\alpha' = \delta'\gamma' = 0.$

We denote by $P(i)$ and $I(i)$ the indecomposable projective and injective modules corresponding to the vertex i, respectively. The indecomposable projective A-modules and B-modules are displayed, respectively.

$P(1)$	$P(2)$	$P(3)$	$P(4)$		$P(1')$	$P(2')$	$P(3')$	$P(4')$
1	2	3	4		$1'$	$2'$	$3'$	$4'$
2	3	4	1		$2'$	$1'$	$4'$	$3'$
3	1	1			$1'$		$3'$	

The indecomposable injective A-modules and B-modules are given as follows.

$I(1)$	$I(2)$	$I(3)$	$I(4)$		$I(1')$	$I(2')$	$I(3')$	$I(4')$
3	1	1	3		$1'$	$1'$	$3'$	$3'$
4	2	2	4		$2'$	$2'$	$4'$	$4'$
1		3			$1'$		$3'$	

Then A-stp $= \mathrm{add}\big(P(1) \oplus P(3)\big)$, B-stp $= \mathrm{add}\big(P(1') \oplus P(3')\big)$ and ν-dom.dim$(A) = \nu$-dom.dim$(B) = 2$. The Frobenius parts Λ and Γ of A and B are given by the quivers with relations, respectively.

$$\Lambda: \quad 1\bullet \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} \bullet 3 \qquad\qquad \Gamma: \quad 1'\bullet \circlearrowleft \alpha' \quad 3'\bullet \circlearrowleft \gamma'$$

$$\alpha\gamma = \gamma\alpha = 0, \qquad\qquad \alpha'^2 = \gamma'^2 = 0.$$

It follows from [24, Theorem 2.10] (see Lemma 3.13(2) below) that both A and B are stably equivalent to the path algebra C of the quiver

$$C: \quad 1\bullet \longrightarrow \overset{2}{\bullet} \longrightarrow \bullet 5 \qquad 3\bullet \longrightarrow \overset{4}{\bullet} \longrightarrow \bullet 6 \,.$$

Thus A and B are stably equivalent, and so are Λ and Γ by Theorem 1.3. Now, we consider the stably equivalent algebras A and C. Clearly, ν-dom.dim$(C) = 0$ and the Frobenius part of C is 0. Thus the Frobenius part Λ of A is not stably equivalent to the Frobenius part of C. This shows that the assumption on ν-dominant dimensions on Artin algebras in Theorem 1.3 cannot be omitted. Observe that dom.dim$(C) = 1$. This shows that the ν-dominant dimensions in Theorem 1.3 cannot be weakened to dominant dimensions either.

3.3 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4.

In [24, Theorem 2.10], Martínez-Villa showed that any Artin algebra with nodes is stably equivalent to an Artin algebra without nodes. The process of removing nodes runs precisely as follows. Suppose that A is an Artin algebra with nodes. Let $\{S(1), S(2), \cdots, S(n)\}$ be a complete set of non-isomorphic simple A-modules. Suppose that $P(i) = Ae_i$ has the top $S(i)$ with $e_i^2 = e_i \in A$ for $1 \leq i \leq n$. We may assume that $\{S(1), \cdots, S(m)\}$ is a complete set of nodes of A with $m \leq n$. Set $S := \bigoplus_{i=1}^{m} S(i)$. Let I be the trace of S in A. Then $I \in \mathrm{add}(_A S)$. By the definition of nodes, $S \in \mathrm{add}(\mathrm{soc}(_A A))$ and $S \in \mathrm{add}(_A I)$. Thus $\mathrm{add}(_A I) = \mathrm{add}(_A S)$. Clearly, $I^2 = 0$ and $\mathrm{rad}(A)I = 0$. Let $J := \mathrm{ann}_l(I)$ be the left annihilator of I. Then $\mathrm{rad}(A) \subseteq J$ and A/J is semisimple. Since $_A I$ has only composition factors $S(i)$ for $1 \leq i \leq m$, we have $e_i I \neq 0$ for $1 \leq i \leq m$ and $e_j I = 0$ for $m+1 \leq j \leq n$. This yields $\mathrm{add}(_A A/J) = \mathrm{add}(_A S)$. Note that I is a two-sided ideal of A and $JI = 0$. Thus I is an (A/J)-(A/I)-bimodule. Let A' be the triangular matrix algebra

$$A' := \begin{pmatrix} A/I & 0 \\ I & A/J \end{pmatrix}.$$

The following lemma describes some common properties of A and A'.

Lemma 3.13. (1) *The triangular matrix Artin algebra A' has no nodes.*
(2) *A and A' are stably equivalent.*
(3) *A and A' have the same numbers of non-isomorphic, non-projective simples.*
(4) *If A is Frobenius-finite, then so is A'.*

Proof. The first two statements are taken from [24, Theorem 2.10], while we prove (3) and (4).

(3) As is known, A'-modules can be identified with triples (X, Y, f), where X is an A/I-module, Y is an A/J-module and $f : I \otimes_{A/I} X \to Y$ is a homomorphism of A/J-modules. It follows from $I^2 = 0$ that $I \subseteq \mathrm{rad}(A)$. Thus simple A-modules coincide with simple A/I-modules, and therefore A and A/I have the same number of non-isomorphic simple modules. Note that the projective cover of an A/I-module X is of the form P/IP with P being a projective cover of the A-module $_A X$. Obviously, the simple A'-modules are either of the form $(T, 0, 0)$, where T is a simple A-module, or of the form $(0, T', 0)$, where T' is a simple A/J-module. The indecomposable projective A'-modules are either of the form $\tilde{P} := (P/IP, I \otimes_{A/I} P/IP, \mathrm{id})$ with P an indecomposable projective A-module or of the form $(0, T', 0)$ with T' an indecomposable projective A/J-module. Thus $(0, T', 0)$ is a projective simple A'-module, and so the indecomposable non-projective simple A'-modules are of the form $(T, 0, 0)$, where T is a simple A-module.

We prove that A and A' have the same number of non-isomorphic, non-projective simples. Indeed, take a simple A-module T, then $IT = 0$ and $(T, 0, 0)$ is a simple A'-module. If T is a projective A-module, then T is also a projective A/I-module, and therefore $(T, 0, 0)$ is a projective A'-module. Thus $(T, 0, 0)$ is a projective simple A'-module. Suppose that T is not a projective A-module. Let $P(T)$ be a projective cover of $_A T$. Then $P(T) \not\simeq T$. If $IP(T) = 0$, then $P(T)$ is a projective cover of $_{A/I} T$. Thus $(P(T), 0, 0)$ is a projective cover of $(T, 0, 0)$ and $(T, 0, 0)$ is not a projective A'-module. If $IP(T) \neq 0$, then it follows that $I \otimes_{A/I} \big(P(T)/IP(T)\big) \simeq I \otimes_A \big(P(T)/IP(T)\big) \simeq I \otimes_A (A/I) \otimes_A P(T) \simeq I \otimes_A P(T) \simeq IP(T) \neq 0$. Thus the A'-module $\big(P(T)/IP(T), I \otimes_{A/I} P(T)/IP(T), \mathrm{id}\big)$ is a projective cover of $(T, 0, 0)$. This implies that $(T, 0, 0)$ is not projective. Thus A and A' have the same numbers of non-isomorphic, non-projective simples.

(4) Suppose that $(_{A/I} X, _{A/J} Y, f)$ is an indecomposable A'-module in A'-stp. We show that $_{A/I} X \in A/I$-stp and $_{A/J} Y \in A/J$-stp. Indeed, $(_{A/I} X, _{A/J} Y, f)$ is projective-injective with $\nu_{A'}(_{A/I} X, _{A/J} Y, f) \in A'$-stp. It follows from [6, Proposition 2.5, p.76] that there are two possibilities:

(a) $_{A/J} Y = 0$ and $_{A/I} X$ is an indecomposable projective-injective A/I-module with $I \otimes_{A/I} X = 0$;
(b) $_{A/I} X = 0$ and $_{A/J} Y$ is an indecomposable projective-injective A/J-module with $\mathrm{Hom}_{A/J}(I, Y) = 0$.

Suppose (a) holds. Let $T_0 := \mathrm{top}(_{A/I} X)$. Then $\nu_{A/I}(X)$ is an injective envelope of $_{A/I} T_0$. By [6, Proposition 2.5, p.76], we see that $_{A'}(T_0, 0, 0)$ is a simple A'-module, that $_{A'}(X, 0, 0)$ is a projective cover of $_{A'}(T_0, 0, 0)$, and that $_{A'}(\nu_{A/I}(X), 0, 0)$ is an injective envelope of $_{A'}(T_0, 0, 0)$. Thus $\nu_{A'}(_{A/I} X, 0, 0) \simeq (\nu_{A/I}(X), 0, 0) \in A'$-stp.

This implies that $\nu^i_{A/I}(X)$ is projective-injective for all $i \geq 0$, and $X \in A/I$-stp. Similarly, if (b) holds, then $Y \in A/J$-stp.

Let S be the direct sum of all non-isomorphic nodes of A. Since I is the trace of S in A and J is the left annihilator of I, we have $\mathrm{add}(_AI) = \mathrm{add}(_AS) = \mathrm{add}(_AA/J)$. Thus $\mathrm{Hom}_A(I,Z) = \mathrm{Hom}_{A/J}(I,Z) \neq 0$ for any A/J-module $Z \neq 0$. Hence we can assume that $\{(X_1,0,0),\cdots(X_r,0,0)\}$ is a complete set of all non-isomorphic indecomposable modules in A'-stp for some natural number r and $X_i \in A/I$-stp with $I \otimes_{A/I} X_i = 0$ for $1 \leq i \leq r$. Since each indecomposable projective A/I-module is of the form P/IP for some indecomposable projective A-module P, we assume $X_i \simeq P_i/IP_i$ for some indecomposable projective A-module P_i. Note that $I \otimes_{A/I} (Q/IQ) \simeq I \otimes_A (Q/IQ) \simeq I \otimes_A (A/I) \otimes_A Q \simeq I \otimes_A Q \simeq IQ$ for each projective A-module Q. It follows from $I \otimes_{A/I} X_i = 0$ that $IP_i = 0$, and therefore $X_i \simeq P_i$ as A/I-modules and $\mathrm{soc}(_AP_i)$ is a simple A-module. Since P_i is a projective A-module, the trace of S in P_i is equal to IP_i. It then follows from $IP_i = 0$ that $\mathrm{soc}(_AP_i)$ has no nodes as its direct summands for $1 \leq i \leq r$. Set $U := \bigoplus_{i=1}^r P_i$. Then $\mathrm{soc}(_AU)$ has no nodes as its direct summands. Since $\nu_{A'}(U,0,0)$ is ν-stably projective and $\nu_{A'}(U,0,0) \simeq (\nu_{A/I}(U),0,0)$, there hold $U \simeq \nu_{A/I}(U)$ and $\mathrm{top}(_{A/I}U) \simeq \mathrm{soc}(_{A/I}U)$ as A/I-modules. Thus $\mathrm{top}(_AU)$ is isomorphic to $\mathrm{soc}(_AU)$ and has no nodes as its direct summands. In particular, $\mathrm{Hom}_A(U,I) = 0$. Applying $\mathrm{Hom}_A(U,-)$ to the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

of A-A-bimodules, we get the exact sequence of A^{op}-modules

$$0 \longrightarrow \mathrm{Hom}_A(U,I) \longrightarrow \mathrm{Hom}_A(U,A) \longrightarrow \mathrm{Hom}_A(U,A/I) \longrightarrow 0.$$

It follows from $\mathrm{Hom}_A(U,I) = 0$ that $\mathrm{Hom}_A(U,A) \simeq \mathrm{Hom}_A(U,A/I)$. Clearly, $\mathrm{Hom}_A(U,A/I) = \mathrm{Hom}_{A/I}(U,A/I)$ as A^{op}-modules. Thus $D\mathrm{Hom}_A(U,A) \simeq D\mathrm{Hom}_{A/I}(U,A/I)$ as A-modules. As $\nu_{A/I}(U) = D\mathrm{Hom}_{A/I}(U,A/I) \simeq U$ as A-modules and $D\mathrm{Hom}_A(U,A) \simeq U$ as A-modules, we get $U \in A$-stp. Let Λ be the Frobenius part of A, and let Λ' be the Frobenius part of A'. Then $\mathrm{End}_A(U)$ is of the form $f\Lambda f$ for an idempotent $f \in \Lambda$, and

$$\Lambda' := \mathrm{End}_{A'}((U,0,0)) \simeq \mathrm{End}_{A/I}(U) \simeq \mathrm{End}_A(U).$$

If A is Frobenius-finite, then $\mathrm{End}_A(U)$ is representation-finite, and therefore A' is Frobenius-finite. $\square$

Lemma 3.14. *Let A and B be stably equivalent Artin algebras, and let Λ and Γ be the Frobenius parts of A and B, respectively.*

(1) If A and B have no semisimple direct summands, then so do Λ and Γ.

(2) Suppose that A and B have no nodes. Then Λ and Γ are stably equivalent. If, in addition, Λ and Γ have the same number of non-isomorphic, non-projective simples, then so do A and B.

(3) Suppose that ν-dom.dim$(A) \geq 1$ and ν-dom.dim$(B) \geq 1$. If one of Λ and Γ is a Nakayama algebra, then A and B have the same number of non-isomorphic, non-projective simples.

Proof. Let $X \in A$-mod and $Y \in B$-mod such that A-stp $= \mathrm{add}(_AX)$ and B-stp $= \mathrm{add}(_BY)$, and let $\Lambda := \mathrm{End}(_AX)$ and $\Gamma := \mathrm{End}_B(Y)$. Then Λ and Γ are the Frobenius parts of A and B, respectively, and therefore they are self-injective Artin algebras.

(1) We show that if Λ has semisimple direct summands then so does A. Indeed, without loss of generality, we may assume that A is a basic algebra and $_AX$ is a basic A-module. Then Λ is a basic algebra. Since Λ has semisimple direct summands, there is a nonzero central idempotent e of Λ such that $e\Lambda e$ is semisimple and $\Lambda = e\Lambda e \times (1-e)\Lambda(1-e)$. In particular, $(1-e)\Lambda e = e\Lambda(1-e) = 0$. Nota that $e\Lambda e$ is basic. It follows from the Wedderburn-Artin theorem that $e\Lambda e$ is isomorphic to a product of finitely many division rings. Let e_0 be a primitive idempotent of Λ with $e_0 \in e\Lambda e$. Then $e_0\Lambda e_0$ is a division ring and $\Lambda = e_0\Lambda e_0 \times (e - e_0)\Lambda(e - e_0) \times (1-e)\Lambda(1-e)$. Particularly, $(1-e_0)\Lambda e_0 = e_0\Lambda(1-e_0) = 0$. Since the evaluation functor $\mathrm{Hom}_A(X,-) : A$-mod $\to \Lambda$-mod induces an equivalence $\mathrm{add}(_AX) \simeq \Lambda$-proj of additive categories, there is an indecomposable summand X_0 of $_AX$ such that $\mathrm{Hom}_A(X,X_0) \simeq \Lambda e_0$. Then $\mathrm{End}_A(X_0) \simeq e_0\Lambda e_0$ is a divisor ring, $\mathrm{Hom}_A(X/X_0,X_0) = (1-e_0)\Lambda e_0 = 0$, and $\mathrm{Hom}_A(X_0,X/X_0) = e_0\Lambda(1-e_0) = 0$. As $_AX \in A$-stp is basic,

$\text{top}(_A X) \simeq \text{soc}(_A X)$. Thus $\text{top}(_A X_0) \simeq \text{soc}(_A X_0)$. Since $\text{End}_A(X_0)$ is a division ring, X_0 must be a simple A-module in A-prinj. Then $\text{Hom}_A(X_0, P) = \text{Hom}_A(P, X_0) = 0$ for any indecomposable projective A-module P which is not isomorphic to X_0. Thus $A \simeq \text{End}_A(X_0) \times \text{End}_A(A/X_0)$. In particular, $\text{End}_A(X_0)$ is a semisimple direct summand of A.

(2) Since we concern only non-projective simple modules, we may assume that A and B have no semisimple direct summands. It follows from [26, Theorem 2.6] that Λ and Γ are stably equivalent.

Assume further that Λ and Γ have the same number of non-isomorphic, non-projective simple modules. We show that A and B have the same number of non-isomorphic, non-projective simple modules. Indeed, it follows from [26, Lemma 2.5] which holds true also for Artin algebras, that A and B have the same number of non-isomorphic, non-projective, simple modules whose projective covers are not ν-stably projective. It remains to show that A and B have the same number of non-isomorphic, non-projective, simple modules whose projective covers are ν-stably projective. Note that a projective simple module is not ν-stably projective. Otherwise, it would be a projective-injective simple module, and therefore A and B would have semisimple direct summands. Thus we have to show that A and B have the same number of non-isomorphic simple modules whose projective covers are ν-stably projective. As A-stp $= \text{add}(_A X)$ and B-stp $= \text{add}(_B Y)$, we need to show that Λ and Γ have the same number of non-isomorphic simple modules. Note that Λ and Γ do not have projective simple modules by (1). By assumption, Λ and Γ have the same number of non-isomorphic simple modules. Hence A and B have the same numbers of non-isomorphic, non-projective simple modules.

(3) Without loss of generality, we assume that A and B have no semisimple direct summands. We have to show that A and B have the same numbers of non-isomorphic, non-projective simples. Indeed, due to ν-dom.dim$(A) \geq 1$ and ν-dom.dim$(B) \geq 1$, it follows from Lemma 3.11 that A and B have the same number of non-isomorphic, non-projective, simple modules whose projective covers are not injective. By Lemma 3.10(1), A-stp $= A$-prinj and B-stp $= B$-prinj. It remains to show that A and B have the same number of non-isomorphic, non-projective, simple modules whose projective covers are ν-stably projective. By Theorem 1.3, Λ and Γ are stably equivalent. Assume that one of Λ and Γ is a Nakayama algebra. By [29, Theorem 1.3] which says that if an Artin algebra is stably equivalent to a Nakayama algebra then the two algebras have the same number of non-isomorphic, non-projective simple modules, we deduce that Λ and Γ have the same number of non-isomorphic, non-projective simples. An argument similar to the proof of (2) shows that A and B have the same number of non-isomorphic, non-projective, simple modules whose projective covers are ν-stably projective. Thus A and B have the same numbers of non-isomorphic, non-projective simples. $\square$

A finite-dimensional k-algebra A over a field k is called a *Morita algebra* if A is isomorphic to $\text{End}_H(H \oplus M)$ for H a finite-dimensional self-injective k-algebra and M a finitely generated H-module [21]. If H is symmetric, then the Morita algebra A is called a *gendo-symmetric algebra* [13]. In this case, the Frobenius part of A is Morita equivalent to H. Recently, it is shown that $S_n(c, k)$ is always a gendo-symmetric algebra [33, Theorem 1.1(2)]. An algebra A is a Morita algebra if and only if ν-dom.dim$(A) \geq 2$ by [12, Proposition 2.9].

For $c \in M_n(k)$, we denote by $k[c]$ the unitary subalgebra of $M_n(k)$ generated by c. Let $\varphi : k[x] \to k[c]$ be the surjective homomorphism of algebras, defined by $x \mapsto c$. Then $\text{Ker}(\varphi) = (m_c(x))$ where $m_c(x)$ is the minimal polynomial of c over k, and φ induces an isomorphism $\bar{\varphi} : k[x]/(m_c(x)) \simeq k[c]$ of algebras. Let $A_c := k[x]/(m_c(x))$, and let k^n be the n-dimensional vector space over k consisting of column vectors. Then k^n is naturally a $k[c]$-module, and therefore an A_c-module via $\bar{\varphi}$. By definition, $S_n(c, k)^{\text{op}} \simeq \text{End}_{A_c}(k^n)$. If we write $m_c(x) := \prod_{i=1}^s f_i(x)^{n_i}$ with all $f_i(x)$ pairwise coprime irreducible polynomials and set $B_i := k[x]/(f_i(x)^{n_i})$ for $1 \leq i \leq s$, then it follows from the Chinese remainder theorem that $A_c := k[x]/(m_c(x)) \simeq \prod_{i=1}^s B_i$. Now, we decompose the A_c-module $k^n = \bigoplus_{i=1}^s M_i$ such that M_i is the direct sum of indecomposable direct summands of k^n lying in the block B_i. Then $S_n(c, k)^{\text{op}} \simeq \prod_{i=1}^s \text{End}_{B_i}(M_i)$. Clearly, k^n is a faithful $M_n(k)$-module and $k[c]$ is a subalgebra of $M_n(k)$. Thus k^n is also a faithful $k[c]$-module. This implies that M_i is a faithful B_i-module for $1 \leq i \leq s$. As B_i is a symmetric Nakayama algebra (see [6, Section V.1 Example, pp. 140-141]), we know that M_i is a generator for B_i-mod and $\text{End}_{B_i}(M_i)$ is a gendo-symmetric algebra for $1 \leq i \leq s$. Due to the isomorphisms $S_n(c, k) \simeq S_n(c', k) \simeq S_n(c, k)^{\text{op}}$ as algebras, where c' is the transpose of the matrix c, we see that the gendo-symmetric algebra $S_n(c, k)$ has its Frobenius part Morita equivalent to B_i for $1 \leq i \leq s$. Thus $S_n(c, k)$ is a gendo-symmetric algebra such that its Frobenius part is a symmetric Nakayama algebra.

Proof of Theorem 1.4. (1) Let $c \in M_n(k)$ and $d \in M_m(k)$. Suppose that $S_n(c,k)$ and $S_m(d,k)$ are stably equivalent. Since $S_n(c,k)$ and $S_m(d,k)$ are gendo-symmetric, it follows from [12, Proposition 2.9] that ν-dom.dim$(S_n(c,k)) \geq 2$ and ν-dom.dim$(S_m(d,k)) \geq 2$. Thanks to Theorem 1.3, the Frobenius parts of both $S_n(c,k)$ and $S_m(d,k)$ are also stably equivalent. Note that the Frobenius parts of both $S_n(c,k)$ and $S_m(d,k)$ are Nakayama algebras. It follows from Lemma 3.14(3) that $S_n(c,k)$ and $S_m(d,k)$ have the same numbers of non-isomorphic, non-projective simples.

(2) Assume that A and B are Artin k-algebras over a commutative Artin ring k. Given a stable equivalence between A and B, we get a stable equivalence between A' and B' both of which have no nodes. Let Λ' and Γ' be the Frobenius parts of A' and B', respectively. Then Λ' and Γ' are stably equivalent by Lemma 3.14(2).

Now, assume that k is an algebraically closed field and that A is Frobenius-finite. Then A' is Frobenius-finite by Lemma 3.13(4), that is, Λ' is representation-finite and therefore Γ' is representation-finite. Since Auslander-Reiten conjecture holds true for a stable equivalence between representation-finite k-algebras over an algebraically closed field k (see [25, Theorem 3.4]), Λ' and Γ' have the same number of non-isomorphic, non-projective simple modules. By Lemma 3.14(2), A' and B' have the same number of non-isomorphic non-projective simple modules, and therefore A and B have the same number of non-isomorphic non-projective simple modules by Lemma 3.13(3). $\square$

The following result is an immediate consequence of Theorems 1.3 and 1.4(2). Here algebras considered may have nodes.

Corollary 3.15. *Every stable equivalence of Morita k-algebras over a field k induces a stable equivalence of their Frobenius parts. In particular, if A and B are stably equivalent Morita algebras over an algebraically closed field and if one of A and B is Frobenius-finite, then A and B have the same number of non-isomorphic, non-projective simples.*

Finally, we suggest the following conjecture, though derived equivalences do not have to preserve the delooping levels of algebras in general.

Conjecture. If A and B are derived equivalent noetherian rings, then $\mathrm{del}(A) < \infty$ if and only if $\mathrm{del}(B) < \infty$.

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