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# On Frobenius extensions of the centralizer matrix algebras

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**Abstract.** We establish a characterization of when a matrix algebra is a Frobenius extension of its centralizer subalgebra.

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**Keywords.** Centralizer algebras, Frobenius extension, Matrix algebra.

Let $R$ be a (unitary associative) ring and $C$ be a nonempty set of $R$. The *centralizer* of $C$ in $R$ is a subring of $R$ defined by

$$S(C,R) := \{r \in R \mid rc = cr \text{ for all } c \in C\}.$$

If $C = \{c\}$ is a singleton set, then $S(c,R) := S(\{c\},R)$ is called a *principal centralizer* ring. We recommend [5,6] as basic references for combinatoric characterizations, representation theory and homological properties of principal centralizer rings.

Let $R$ and $S$ be two rings. Recall that a bimodule ${}_SP_R$ is a *Frobenius bimodule* if both ${}_RP$ and $P_S$ are finitely generated projective modules, and there is an $R$-$S$-bimodule isomorphism

$$\mathrm{Hom}_S(P,S) \cong \mathrm{Hom}_{R^{op}}(P,R).$$

An extension $S \subseteq R$ of rings is called a *Frobenius extension* if ${}_SR_R$ is a Frobenius bimodule.

For a ring $R$ and a positive integer $n$, $M_n(R)$ denotes the full matrix ring of all $n \times n$ matrices over $R$. It is shown in [6, Theorem 1.1.(1)] that if $R$ is a field, then $S(c, M_n(R)) \subseteq M_n(R)$ is always a Frobenius extension for any $c \in M_n(R)$. In [5], Xi and Zhang considered the following general question.

**Question 1** ([5, Question 4.10 (2)]). *Let $R$ be a ring and $n$ be a positive integer. For any $c \in M_n(R)$, is $M_n(R)$ always a Frobenius extension of $S(c, M_n(R))$?*

In general, $S(C,R) \subseteq R$ is not a Frobenius extension even when $R$ is a matrix algebra, see [5, Remark 3.13].

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This paper is devoted to give a characterization of when the matrix algebra over a commutative ring is a Frobenius extension of its centralizer subalgebra.

We start with the following endomorphism ring theorem.

**Proposition 2** ([2, Theorem 2.5 and 2.8]). *Let $P$ be a progenerator over a ring $R$ and $S$ be a subring of $\operatorname{End}_R(P)$. Then $\operatorname{End}_{R^{op}}(P)$ is a Frobenius extension of $S$ if and only if ${}_SP_R$ is a Frobenius bimodule.*

**Theorem 3.** *Let $R$ be a commutative ring, $P$ be a progenerator over $R$, and $A = \operatorname{End}_R(P)$. Let $C$ be an $R$-subalgebra of $A$. Then $A$ is a Frobenius extension of $S(C, A)$ if and only if*

(1) *$P$ is a generator as a left $C$-module, and*
(2) *$C$ is a symmetric Frobenius extension of $R$ (i.e., $\operatorname{Hom}_R({}_CC_C, R) \cong {}_CC_C$).*

*Proof.* Let $S = S(C, A)$ and $B = \operatorname{End}_C(P) \cong S^{op}$.

If $C$ is a symmetric Frobenius $R$-algebra, then

$$\operatorname{Hom}_C(P, C) \cong \operatorname{Hom}_C(P, \operatorname{Hom}_R(C, R)) \cong \operatorname{Hom}_R(C \otimes_C P, R) \cong \operatorname{Hom}_R(P, R)$$

as $B$-$C$-bimodule. Since ${}_CP$ is a generator, by a result of Morita (see [3, Proposition 18.17]), $P$ is a projective right $B$-module, and $\operatorname{Hom}_C(P, C) \cong \operatorname{Hom}_{B^{op}}(P, B)$ as $B$-$C$-bimodules. Since $B \cong S^{op}$, ${}_SP_R$ is a Frobenius bimodule. By Proposition 2, $A$ is a Frobenius extension of $S$.

Now assume that $A$ is a Frobenius extension of $S$. By Proposition 2, ${}_SP_R$ is a Frobenius bimodule, so is ${}_RP_B$. Since $V$ is a projective right $B$-module, by a result of Morita, ${}_CP$ is a generator and $C \cong P \otimes_B \operatorname{Hom}_{B^{op}}(P, B) \cong \operatorname{Hom}_{B^{op}}(P, P)$ as $C$-$C$-bimodules. Then

$$\begin{aligned}
\operatorname{Hom}_R(C, R) &\cong \operatorname{Hom}_R(P \otimes_B \operatorname{Hom}_{B^{op}}(P, B), R) \\
&\cong \operatorname{Hom}_R(P \otimes_B \operatorname{Hom}_R(P, R), R) && \text{since}_R P_B \text{is a Frobenius bimodule} \\
&\cong \operatorname{Hom}_{B^{op}}(P, \operatorname{Hom}_R(\operatorname{Hom}_R(P, R), R)) \\
&\cong \operatorname{Hom}_{B^{op}}(P, P) \cong C
\end{aligned}$$

as $C$-$C$-bimodules. So $C$ is a symmetric Frobenius $R$-algebra. □

**Corollary 4.** *Let $\Bbbk$ be a field, and $C$ be a commutative subalgebra of $M_n(\Bbbk)$. Then $M_n(\Bbbk)$ is a Frobenius extension of $S(C, M_n(\Bbbk))$ if and only if $C$ is a Frobenius algebra.*

*Proof.* Notice that a commutative Frobenius algebra $C$ is always symmetric, and that any faithful $C$-module $M$ is always a generator [1, Theorem 59.3]. Hence the conclusion follows immediately from Theorem 3. □

For any $c \in M_n(\Bbbk)$, since $\Bbbk[c]$ is a commutative Frobenius algebra (see [4, Corollary 4.36]), $M_n(\Bbbk)$ is a Frobenius extension of $S(c, M_n(\Bbbk))$ by Corollary 4.

If $R$ is just a commutative ring but not a field, then there exists an example such that $M_n(R)$ is not a Frobenius extension of $S(c, M_n(R))$.

*Example 5.* Let $R = \Bbbk[X]/(X^2)$, $x = X + (X^2)$, and $c = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in M_2(R)$. Let $C$ be the $R$-subalgebra of $M_2(R)$ which is generated by $c$. It is clear that

$C \cong \Bbbk[X,Y]/(X^2,Y^2,XY)$ which is not a Frobenius algebra. Then $M_2(R)$ is not a Frobenius extension of $S(C, M_2(R)) = S(c, M_2(R))$ by Theorem 3.

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